Embedding of a truncated vector lattice into its universal completion

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The Framework

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Definition

A unary operation $*$ on the positive cone L^+ of L is called a **truncation** if

1. $f^* \wedge g = f \wedge g^*$ for all $f, g \in L^+$, and
2. if $f \in L^+$ and $(nf)^* = nf$ for all $n \in \{1, 2, ...\}$ then $f = 0$.

- By a **truncated vector lattice** we mean a (real) vector lattice L along with a truncation

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$$\text{if } f \in L^+ \text{ and } f^* = 0 \text{ then } f = 0$$

are referred to as **weakly truncated vector lattices**.

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THEOREM (BALL, 2014)

For any weakly truncated vector lattice L, a locally compact Hausdorff space X can be found such that L is (lattice isomorphic with) a vector lattice of functions in $C^{\infty}(X)$ *and the truncation on L is provided by meet with the constant function one.*

Theorem (Boulabiar-H, 2021)

For any truncated vector lattice L, there exist a locally compact Hausdorff space X and a clopen set Y in X such that L is (lattice isomorphic with) a vector lattice of functions in $C^{\infty}(X)$ and

$$f^{*} = 1_{Y} \wedge f$$

for all $f \in L^{+}$, where 1_{Y} denotes the characteristic function on Y.

The Problem

- Both approaches to get such representations are topological in nature and quite involved. The main purpose was to obtain (in a more direct way) algebraic versions of these results by embedding the truncated vector lattice L under consideration into its universal completion L^u and to identify an element e in L^u that serves as a truncation unit for L, that is to say,

$$f^* = e \wedge f \text{ for all } f \in L^+$$

The set of all fixed points of the truncation of L is denoted by Φ. Observe that

$$\Phi = \left\{ f^* : f \in L^+ \right\}.$$

Lemma

The set Φ has a supremum in L^u.

Proof.

The key idea of the proof is the inequality

$$f - f^* \leq (f - g^*)^+ \text{ for all } f, g \in L^+.$$

As L^u is Dedekind complete, it suffices to show that Φ is bounded from above. Let $f \in L$ with $0 < f$. By the condition (2) in the definition of truncated vector lattices, we can find $n \in \{1, 2, ...\}$ such that $nf > (nf)^*$. Put $g = nf - (nf)^*$ and pick $h \in L^+$. Using the inequality $f - f^* \leq (f - g^*)^+$ for all $f, g \in L^+$,we can write

$$0 < g = nf - (nf)^* \leq (nf - h^*)^+, \text{for every } h \in L^+.$$

PROOF.

Hence,

$$0 < g \leq (nf - h)^+ \text{ for all } h \in \Phi,$$

which means that Φ is a dominable set of L in the sense of definition above.Taking into consideration Theorem 7.14 in *Locally Solid Riesz Spaces with Applications to Economics (a subset A of the positve cone of an archimedean complete Riesz space M* is dominable if and only if it is order bonded in *M)*, we infer that Φ has an upper bound L^u, as desired. □

- The supremum of Φ in L^u is denoted by e. Hence,

$$e = \sup_{L^u} \left\{ f^* : f \in L^+ \right\} = \min \left\{ u \in L^u : f^* \leq u \text{ for all } f \in L^+ \right\}.$$

Answer for the General case

Theorem (Boulabiar-H, 2022)

Let L be a truncated vector lattice. There exists a component e of a distinguished weak unit of L^u such that

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Proof.

If $f \in L^+$ then $f^* \in \Phi$, so

$$\begin{aligned} f^* &\leq e \wedge f = \sup_{L^u} \{g^* : g \in L_+\} \wedge f \\ &= \sup_{L^u} \{g^* \wedge f : g \in L_+\} = \sup_{L^u} \{g \wedge f^* : g \in L_+\} \leq f^*. \end{aligned}$$

This allows us to conclude. □

Answer for the weak case

Corollary

Assume that L is a weakly truncated vector lattice. Then there exists a weak unit w in L^u such that

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Proof.

We claim that e is a weak unit in L^u. Let $u \in L^u$ such that $e \wedge u = 0$ and assume, by contradiction, that $u > 0$. As L is order dense in L^u , there exists $f \in L$ such that $0 < f \leq u$. But then

$$0 \leq f^* = e \wedge f \leq e \wedge u = 0.$$

We derive that $f^* = 0$ and so $f = 0$ because L is weakly truncated. Contradiction ! □

What we are concerned with here is that e is a component of some weak unit w in L^u. Now, by the classical Theorem of Maeda-Ogasawara, there exists a unique (up to homeomorphism) Stonean (i.e., extremally disconnected, compact, and Hausdorff) space X such that $C^\infty(X)$ is an a lattice isomorphic to L^u, the universal completion of L.

Moreover, a lattice isomorphism between L^u and $C^\infty(X)$ can be constructed so that the above weak element w can be identified with the constant function 1 in $C^\infty(X)$ and, consequently, e becomes a characteristic function of some clopen set Y in X. It is also worth mentioning that the space $C^\infty(X)$ here is a vector lattice, which is not the case of the space of almost-finite extended-real continuous valued functions that serves to get the formentioned representations.

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