The Alexandroff unitization of a Lattice ordered algebra with a truncation

Presented By:

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Plan

1. Truncated vector lattice
2. Alexandroff unitization of a truncated vector lattice.
3. Universal properties of the Alexandroff unitization of a truncated vector lattice.
4. Unitization of lattice ordered algebra with a truncation

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Definition (Ball, (2014))

▷ A ***truncation*** on a vector lattice T is a function that takes each positive element $x \in T$ to a positive element $\tau(x) \in T$ and has the following properties.

(τ_1) $x \wedge \tau(y) \leq \tau(x) \leq x$ *for all* $0 \leq x, y \in T$.

(τ_2) *If* $0 \leq x \in T$ *and* $\tau(x) = 0$ *then* $x = 0$.

(τ_3) If $x \in T^+$ and $nx = \tau(nx)$ for all $n \in \{1, 2, ...\}$, then $x = 0$.

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Definition

A truncated vector lattice U is said to be ***unital*** if there is an element u such that the truncation is provided by meet with u, i.e.,

$$\tau(x) = u \wedge x \quad \text{for all } 0 \leq x \in U.$$

Such an element u is called a ***truncation unit*** of U.

Furthermore, a truncated vector lattice need not be unital as the following example shows.

Example

Let $E = C([0,1])$ the vector lattice of all continuous real-valued functions on the real interval $[0,1]$ and put

$$T = \Big\{ f \in E : f(0) = 0 \Big\}.$$

Obviously, T is a vector sublattice of E. Also, a truncation can be defined on T by

$$\tau(f)(x) = \min\{f(x), 1\}, \quad \textit{for all } f \in T^{+} \text{ and } x \in [0,1].$$

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Using a non-constructive representation by functions (ZFC), ***Ball*** *proved that for any truncated vector lattice T, there exists a vector lattice ϵT with a weak order unit e such that*

1. *T is a vector sublattice of ϵT, and*
2. *$\tau(x) = x \wedge e$ for all $x \in T^+$.*

We introduce the notion of unitizations of a truncated vector lattice next.

Definition

A unital vector lattice ϵT *with truncation unit* $u \geq 0$ *is called a* ***unitization*** *of a truncated vector lattice* (T, τ) *if the following conditions hold.*

1. T is a vector sublattice of ϵT, and
2. $\tau(x) = u \wedge x$ for all $0 \leq x \in T$.

For instance, any unital vector lattice is a unitization of itself. To set these ideas down we give another example.

Example

The real vector lattice of all continuous real-valued functions on $\mathbb{R}$ that vanish at infinity is denoted by $C_0(\mathbb{R})$. Clearly, $C_0(\mathbb{R})$ is a truncated vector lattice with respect to the truncation defined by

$$\tau(x)(r) = \min\{x(r), 1\} \quad \text{for all } x \in C_0(\mathbb{R}) \text{ and } r \in \mathbb{R}.$$

Obviously, the vector lattice $C(\mathbb{R})$ of all continuous real-valued functions on $\mathbb{R}$ is a unitization of $C_0(\mathbb{R})$. Analogously, the vector lattice $C^*(\mathbb{R})$ of all bounded functions in $C(\mathbb{R})$ is another unitization of $C_0(\mathbb{R})$.

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- $T \oplus \mathbb{R}$ has 1 as a weak unit.
- $\tau(x) = x \wedge 1$ for all $x \in T^+$.
- T is an ℓ-ideal in $T \oplus \mathbb{R}$.
- $T \oplus \mathbb{R}$ is a unitization of T.

The vector lattice $T^* = T \oplus \mathbb{R}$ is called the ***Alexandroff unitization*** of T.

How does a truncated vector lattice sit in its Alexandroff unitization ?

Definition

The disjoint complements T^d of T in T^* is given by

$$T^d = \{v \in T^* : |v| \wedge |x| = 0 \text{ for all } x \in T\}.$$

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Let T be a truncated vector lattice. Then the following assertions hold.

1. *T contains at most one truncation unit.*
2. *If T is not unital then $T^d = \{0\}$.*
3. *If T is unital with truncation unit u, then*

$$T^d = \mathbb{R}(1-u) = \{\alpha(1-u) : \alpha \in \mathbb{R}\}.$$

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Corollary

Let T be a truncated vector lattice. Then the following assertions hold.

1. *T is dense in T^* if and only if T is not unital.*
2. *$T \oplus T^d = T^*$ if and only if T is unital.*

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- By a ***truncation homomorphism*** we mean a linear map $S \xrightarrow{\omega} E$ which is a lattice homomorphism and *perseveres truncations*, i.e.,

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- By a ***truncation homomorphism*** we mean a linear map $S \xrightarrow{\omega} E$ which is a lattice homomorphism and *perseveres truncations*, i.e.,

$$\omega(\tau(x)) = \tau(\omega(x)) \text{ for all } x \in S^+.$$

- On the other hand, let U and V be unital (truncated) vector lattices with truncation units u and v, respectively. A linear map $U \xrightarrow{\omega} V$ is called a ***unital lattice homomorphism*** if ω is a lattice homomorphism with $\omega(u) = v$.

• Any unital lattice homomorphism on unital is a truncation homomorphism. However, truncation homomorphism on unital (truncated) vector lattice is a lattice homomorphism but need not be unital

Example

1. $$f:(\mathbb{R},1)\longrightarrow(\mathbb{R},1)$$
$$x\longmapsto 0$$

2. $$f:(\mathbb{R},1)\longrightarrow\left(\mathbb{R}^2,(1,1)\right)$$
$$x\longmapsto(x,0)$$

Theorem (Boulabiar, Hafsi, and M, (2018))

Let T be a truncated vector lattice. Then T^* is the unique (up to a unital lattice isomorphism that leaves T pointwise fixed) unitization of T such that, for every unital vector lattice U, any truncation homomorphism $T \xrightarrow{f} U$ extends uniquely to a unital lattice homomorphism $T^* \xrightarrow{f^*} U$.

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ & \searrow & \uparrow_{f^*} \\ & & T^* \end{array}$$

Theorem (Boulabiar, Hafsi, and M, (2018))

Let T be a non-unital truncated vector lattice and U be a unital vector lattice. Then T^* is the unique (up to a unital lattice isomorphism that leaves T pointwise invariant) unitization of T such that any injective truncation homomorphism $T \overset{f}{\longrightarrow} U$ extends uniquely to an injective unital lattice homomorphism $T^* \overset{f^*}{\longrightarrow} U$.

Corollary

Let T be a non-unital vector lattice. Then T^* is the smallest unitization of T, i.e., if ϵT is another unitization of T then T^* can be embedded in ϵT as a unital vector sublattice.

Plan

Let A be an ℓ-algebra with a truncation τ. Drawing plenty of inspiration from the classical unitization process in Banach Algebra Theory, a natural multiplication can be introduced on the vector lattice $A^* = A \oplus \mathbb{R}$ by putting

$$(x+\alpha)(y+\beta) = xy + \beta x + \alpha y + \alpha\beta, \quad \text{for all } x, y \in A \text{ and } \alpha, \beta \in \mathbb{R}. \qquad (*)$$

It is clair that this multiplication makes A^* into an associative algebra with 1 as identity and A as an algebra ideal. It would seem plausible to think that A^* is even an ℓ-algebra, i.e., the positive cone of A^* is closed under this multiplication. Nevertheless, as the next example shows, such an attractive result cannot be expected without imposing an extra compatibility condition.

Example

The function $\tau : C(\mathbb{R})^+ \to C(\mathbb{R})^+$ defined by

$$\tau(x)(r) = \min\{x(r), 1\}, \quad \text{for all } x \in C(\mathbb{R}) \text{ and } r \in \mathbb{R}$$

is a truncation on $C(\mathbb{R})$. Moreover, it is easily checked that $C(\mathbb{R})$ is an ℓ-algebra under the multiplication given by

$$(xy)(r) = 2x(r)y(r), \quad \text{for all } x, y \in C(\mathbb{R}) \text{ and } r \in \mathbb{R}.$$

Define $x, y \in C(\mathbb{R})$ by

$$x(r) = \cos r \quad \text{and} \quad y(r) = \sin r, \quad \text{for all } r \in \mathbb{R}.$$

Obviously, we have $x^-, y^- \in \tau\left(C(\mathbb{R})^+\right)$ and so $x+1, y+1 \geq 0$ in $\left(C(\mathbb{R})\right)^*$. Furthermore, a simple calculation leads to the equalities

$$(x+1)(y+1) = xy + x + y + 1 \text{ and } (xy + x + y)^-(-\pi/4) = 2.$$

Definition

We call an element x in an ℓ-algebra A with identity $e > 0$ an ***infinitesimal*** if

$$n\,|x| \leq e, \quad \text{for all } n \in \{1, 2, ...\}.$$

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Theorem

Let A be a unital ℓ-algebra such that its identity is simultaneously a weak unit. If A has no non-zero infinitesimals, then A is a semi prime f-algebra.

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Theorem

Let A be an ℓ-algebra with a truncation τ. Then A^* is an ℓ-algebra if and only if A^* is a semiprime f-algebra.

We are in position at this point to state the central result of this section.

Theorem (Boulabiar and M, (2019))

Let A be an ℓ-algebra with a truncation τ. Then A^{*} is an ℓ-algebra if and only if A is a semiprime f-algebra with

$$\tau\left(A^{+}\right)=\left\{x\in A:x^{2}\leq x\right\}.$$

Theorem

Any semiprime Archimedean f-algebra A can be embedded as an f-subalgebra in the unital Archimedean f-algebra $Orth(A)$ of all orthomorphisms on A.

Definition (Ben Amor, Boulabiar, and El Adeb, (2014))

We call the semiprime Archimedean f-algebra A a ***Stone f-algebra*** if

$$\mathrm{id}_A \wedge x \in A, \quad \text{for all } x \in A^+,$$

where id_A denotes the identity map on A (which is the identity of the algebra $\mathrm{Orth}\,(A)$).

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Example

- Any unital Archimedean f-algebra is a Stone f-algebra.
- The f-algebra $C_0(\mathbb{R})$ is a Stone f-algebra.

Definition

We call the ***Stone function*** on the *Stone f-algebra* A the function $\tau : A^+ \rightarrow A^+$ defined by

Definition

We call the ***Stone function*** on the *Stone f-algebra* A the function $\tau : A^+ \to A^+$ defined by

$$\tau(x) = \mathrm{id}_A \wedge x, \quad \text{for all } x \in A^+.$$

Theorem (Boulabiar and M, (2019))

The Stone function on a Stone f-algebra A is the unique truncation τ on A such that the Alexandroff unitization A^* of A is an ℓ-algebra.

Theorem (Boulabiar and M, (2019))

Let A be an Archimedean ℓ-algebra with a truncation τ. Then A^* is an ℓ-algebra if and only if A is a Stone f-algebra and τ is the Stone function.

Thank you for your attention


Merci