Riesz Space Generalizations of recurrence Theorems in Ergodic Theory

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Outline

Classical framework

- The setting: (Ω, Σ, μ) a probability space where:
  - Ω: The collection of all possible states of the dynamical system.
  - Σ: The family of all observable states of the system.
  - μ: Probability of an observable state.
- A measurable map $\tau : \Omega \to \Omega$ is measure preserving if

$$\mu(\tau^{-1}A) = \mu(A) \text{ for } \quad \text{any } A \in \Sigma$$

.

- Probability space $+$ measure preserving transformation $=$ measure preserving system.

Dictionary

(Ω, Σ, μ): probability space	E: Dedekind complete R S
The function 1	The weak order unit e
$A \subset \Omega$: event	P: order projection
χ_A: indicator function	p: component of e
$\mathbb{E}$: expectation, μ: measure	T: conditional expectation

T conditional expectation on E:

- positive
- order continuous
- projection
- $Te = e$
- $R(T)$ is a Dedekind complete Riesz subspace of E.

Setting of Riesz spaces

Conditional expectation preserving system [Homann, Kuo and Watson]

- $$(E, e, T)\text{triple} : : E : \text{Dedekind complete Riesz space}$$
$$e : \text{weak order unit}$$
$$T : \text{ conditional expectation}$$

$$+$$

- S: order continuous Riesz homomorphism on E with

$$Se = e \text{ and } TS = T$$

$$=$$

- (E, e, T, S) conditional expectation preserving system.

Ergodicity

A great part of the theory of measure preserving systems is dedicated to the study of ergodic transformations.

- A measure preserving system $(\Omega, \Sigma, \mu, \tau)$ is ergodic if for all $A \in \Sigma$,

$$\mu(\tau^{-1}A \,\Delta\, A) = 0 \implies \mu(A) = 0 \text{ or } \mu(A) = 1.$$

- $(\Omega, \Sigma, \mu, \tau)$ is ergodic $\iff$ $\forall$ measurable f on Ω,

$$f \circ \tau = f \text{ a.e} \implies f = \text{constant a.e.}$$

$\longrightarrow$ Riesz space setting

Definition (Homann, Kuo and Watson)

The conditional expectation preserving system (E, T, S, e) is said to be ergodic if

$$Sf = f \text{ implies } Tf = f.$$

Recurrence in probability spaces

Definition

Let $(\Omega, \Sigma, \mu, \tau)$ be a measure preserving system. Let $A \in \Sigma$. A point $x \in A$ is said to be recurrent with respect to A if there is $k \in \mathbb{N}$ for which $\tau^k x \in A$.

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Theorem (Poincaré Recurrence Theorem, 1899)

Let $(\Omega, \Sigma, \mu, \tau)$ be a measure preserving system. For each $A \in \Sigma$, almost every point of A is recurrent with respect to A.

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Let $(\Omega, \Sigma, \mu, \tau)$ be a measure preserving system. Let $A \subset \Omega$ be a measurable subset of Ω of positive measure. For $x \in A$,

- $$n_A(x) = \inf\{n \geq 1\ ;\ \tau^n x \in A\}$$

  is called the first recurrence time of x with respect ot A.

- $A_n = \{x \in A\ ;\ n_A(x) = n\}$: the set of points of A recurrent at exactly n iterates for the first time.

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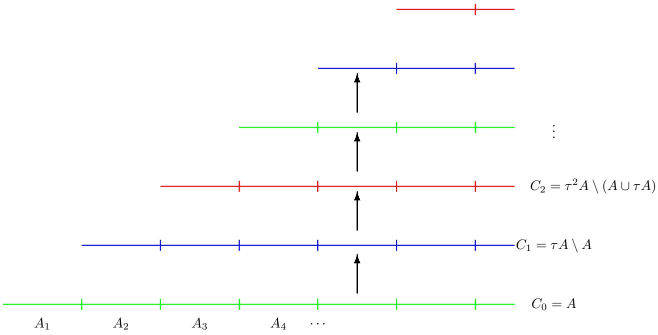
- $A = \bigcup_{n=1}^{\infty} A_n \quad$ (Poincaré).

- $n_A(x) = \sum_{x \in A_k}^{k \in \mathbb{N}} k.$

The action of τ on A is described as follows:



Recurrence in Riesz spaces

The concept of recurrence in a Riesz space will be given in terms of components of a chosen weak order unit.

Definition

Let (E, T, S, e) be a conditional expectation preserving system. Let p and q be components of e with $p \leq q$. We say that p is recurrent with respect to q if there are components $(p_n)_{n\in\mathbb{N}}$ of p so that $p = \bigvee_{n\in\mathbb{N}} p_n$ and $S^n p_n \leq q$ for all $n \in \mathbb{N}$.

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In the case of S bijective,

$$p \text{ is recurrent with respect to } q \iff p \le \bigvee_{n\in\mathbb{N}} S^{-n} q.$$

Poincaré's Recurrence Theorem

Theorem (A-BA-H-M-W,2023)

Let (E, T, S, e) be a conditional expectation preserving system with T strictly positive and S surjective. Let q be a component of e. Then each component p of q is recurrent with respect to q.

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- For $k \in \mathbb{N}$, $q(p,k) = p \wedge (S^{-k}p) \wedge (e - \bigvee_{j=1}^{k-1} S^{-j}p)$: maximal component of p recurrent at exactly k iterates of S.
- $p = \sum_{k=1}^{\infty} q(p,k)$ (Poincaré).
- $n_p = \sum_{k=1}^{\infty} k\, q(p,k)$: first recurrence time for p.

Kac formula

Question: What is the expected value of the time of first return to A?

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Answer: $\frac{1}{\mu(A)}$.

$\frac{1}{\mu(A)} \int_A n_A d\mu$: expected return time to A.

Theorem (Kac Theorem, 1947)

Let $(\Omega, \Sigma, \mu, \tau)$ be an ergodic measure preserving system. Then for each $A \in \Sigma$ with $\mu(A) > 0$, we have

$$\int_A n_A \, d\mu = 1.$$

Kac Formula

The following is a conditional Riesz space analogue of the classical Kac Lemma.

Theorem (A-BA-H-M-W,2023)

Let (E, T, S, e) be an ergodic conditional expectation preserving system, where T is strictly positive, E is T-universally complete (for each increasing net (f_α) in E_+ with (Tf_α) order bounded, (f_α) is order convergent in E) and S is surjective. For each component p of e we have

$$Tn_p = P_{Tp}e.$$

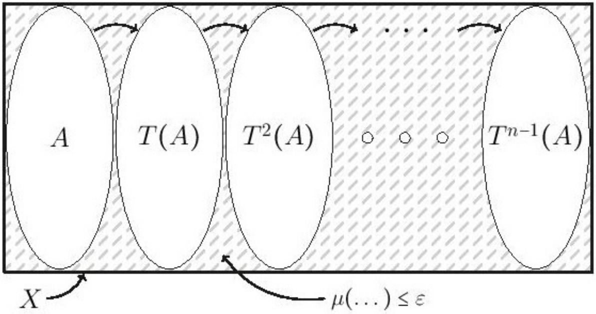
New conditional version of Kac Formula

Consider a complete probability space $(\Omega, \mathcal{F}, \mu)$ and Σ a sub-σ-algebra of $\mathcal{F}$.
Let $\tau : \Omega \to \Omega$ be a surjective measure preserving transformation. Applying the above result gives that,

$$\mathbb{E}[n_A|\Sigma] = \chi_{\{\omega\in\Omega/\mathbb{P}[A|\Sigma](\omega)>0\}} \text{ a.e.}$$

Kakutani Rokhlin Lemma

Question: When a measure preserving dynamical system (X, Σ, μ, T) can be decomposed to an arbitrary high tower of measurable sets and a remainder of arbitrarily small measure?



ϵ-free version of Kakutani-Rokhlin Lemma

Let $(\Omega, \Sigma, \mu, \tau)$ be an ergodic measure preserving system. Let $A \in \Sigma$ with $\mu(A) > 0$ and $n \in \mathbb{N}$. Then there is a set $B \in \Sigma$ such that $B, \tau B, ..., \tau^{n-1}B$ are pairwise disjoint and

$$\mu(\bigcup_{i=0}^{n-1} \tau^i B) \geq 1 - n\mu(A).$$

Theorem (A-M-W)

Let (E, e, T, S) be an ergodic conditional expectation preserving system. Let $n \in \mathbb{N}$ and p be a component of e, then there exists a component q of $P_{Tp}e$ such that $q, Sq, ..., S^{n-1}q$ are pairwise disjoint and

$$T(\bigvee_{i=0}^{n-1} S^i q) \geq (P_{Tp}e - (n-1)Tp)^+$$

- In a probability space (Ω, Σ, μ), μ is said to be nonatomic if for any $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \mu(A)$.
  $\Rightarrow$ Nonatomic spaces ensure the ability to find subsets of arbitrarily small measure
- If $\tau^n = Id$ for some integer n, the transformation τ is said to be periodic, and the period of τ is defined to be the smallest positive n with this property
- aperiodic transformation: transformation whose periodic points form a set of measure 0, that is

$$\mu(\{x \in \Omega \mid \tau^p x = x \text{ for some } p \in \mathbb{N}\}) = 0.$$

If the measure is nonatomic then,
Ergodic $\Longrightarrow$ aperiodic
$\not\Leftarrow$

ϵ-version of Kakutani-Rokhlin Lemma

Under these assumptions, the Kakutani-Rokhlin Lemma can be strengthened to the following statement.

Theorem

If τ is an ergodic invertible measure preserving transformation on a nonatomic probability space (Ω, Σ, μ), then for any natural number n and any $\epsilon > 0$, there exists a measurable set $B \subset \Omega$ such that the sets $B, \tau B, ..., \tau^{n-1} B$ are pairwise disjoint and

$$\mu(\bigcup_{i=0}^{n-1} \tau^i B) > 1 - \epsilon.$$

Let S be a Riesz homomorphism and $0 \neq v$ be a component of e.

- We say that (S, v) is periodic if there is $N \in \mathbb{N}$ so that for all components c of v with $0 \neq c \neq v$ we have that $q(c, k) = 0$ for all $k \geq N$.
- In this case, for all such c we have $c = \bigvee_{k=1}^{N-1} q(c, k)$.
- We say that (S, v) is aperiodic where v is a component of e, if for each $N \in \mathbb{N}$ there exists a component $0 \neq c \neq v$ of v in E so that $q(c, k) \neq 0$ for some $k \geq N$.

Kakutani Rokhlin Lemma in the Riesz space setting

The ϵ- version of the classical Kakutani Rokhlin Lemma can be formulated in Riesz spaces as follows.

Theorem (A,M,W)

Let (E, T, S, e) be an ergodic conditional expectation preserving system with E T-universally complete. If v is a component of e in $R(T)$ with (S, v) aperiodic, $n \in \mathbb{N}$ and $\epsilon > 0$ then there exists a component q of v in E with $(S^i q)_{i=0,\dots,n-1}$ disjoint and

$$T(v - \bigvee_{i=0}^{n-1} S^i q) \leq \epsilon v.$$

Thank you for your attention !

Some bibliography

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