Free Dual Banach Lattices

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Overview

1. Preliminaries
2. $\mathrm{FBL}^{(p)}[E^{**}]$ vs $\mathrm{FBL}^{(p)}[E]^{**}$
3. Free dual Banach lattices

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Free objects

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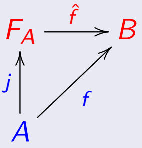

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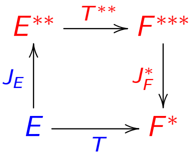

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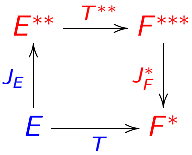

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An operator $T : X \to Y$ between two Banach lattices is called a **lattice homomorphism** if it is linear and preserves the lattice operations.

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Definition

A Banach lattice X is said p**-convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that for any $x_1, \ldots, x_n \in X$ the inequality

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

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The lowest constant M satisfying the above inequality is called the p**-convexity constant of** X, and is denoted by $M^{(p)}(X)$.

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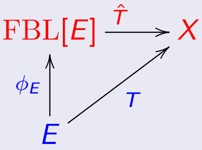

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$$
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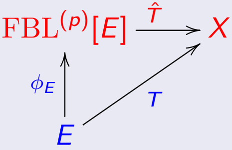

Every Banach lattice is 1-convex with constant 1, so $\mathrm{FBL}^{(1)}[E] = \mathrm{FBL}[E]$.

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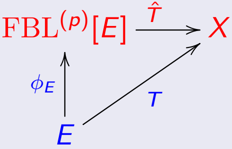

Observation

$\mathrm{FBL}^{(p)}[E]$ exists and is unique!

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*.

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$$\|f\|_p = \sup\left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

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The space $\mathrm{FBL}^{(p)}[E] := \overline{\mathrm{lat}}(\phi_E(E)) \subset H_p[E]$ is a representation of the free p-convex Banach lattice over E.

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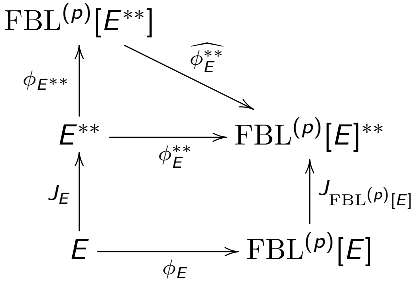

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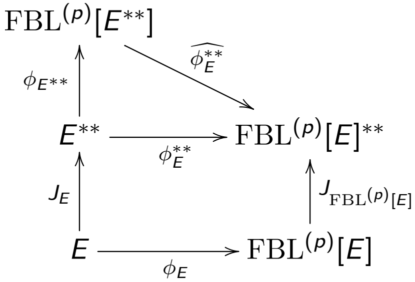

Theorem (GS-Tradacete)

The operator $\widehat{\phi_E^{**}} : \mathrm{FBL}^{(p)}[E^{**}] \to \mathrm{FBL}^{(p)}[E]^{**}$ is an isometric lattice embedding.

Key tool

Theorem (Principle of Local Reflexivity)

Let F be a Banach space. For any finite-dimensional subspaces $U \subset F^{**}$ and $V \subset F^*$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \subset F$ such that $\|S\| \, \|S^{-1}\| \leq 1 + \epsilon$, $x^*(Sx^{**}) = x^{**}(x^*)$ for every $x^* \in V$ and $x^{**} \in U$, and S is the identity on $U \cap J_F(F)$.

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Using this result, we can show that for every $f \in \mathrm{FBL}^{(p)}[E^*]$

$$\sup\left\{\left(\sum_{k=1}^n |f(x_k^{**})|^p\right)^{\frac{1}{p}} : (x_k^{**})_k \subset E^{**},\ \sup_{x^*\in B_{E^*}} \left(\sum_{k=1}^n |x_k^{**}(x^*)|^p\right)^{\frac{1}{p}} \leq 1\right\} =$$

$$\sup\left\{\left(\sum_{k=1}^n |f\circ J_E(x_k)|^p\right)^{\frac{1}{p}} : (x_k)_k \subset E,\ \sup_{x^*\in B_{E^*}} \left(\sum_{k=1}^n |x^*(x_k)|^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

Key tool

Theorem (Principle of Local Reflexivity)

Let F be a Banach space. For any finite-dimensional subspaces $U \subset F^{**}$ and $V \subset F^*$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \subset F$ such that $\|S\|\,\|S^{-1}\| \leq 1+\epsilon$, $x^*(Sx^{**}) = x^{**}(x^*)$ for every $x^* \in V$ and $x^{**} \in U$, and S is the identity on $U \cap J_F(F)$.

Using this result, we can show that for every $f \in \mathrm{FBL}^{(p)}[E^*]$

$$\sup\left\{\left(\sum_{k=1}^n |f(x_k^{**})|^p\right)^{\frac{1}{p}} : (x_k^{**})_k \subset E^{**},\ \sup_{x^*\in B_{E^*}}\left(\sum_{k=1}^n |x_k^{**}(x^*)|^p\right)^{\frac{1}{p}} \leq 1\right\} =$$

$$\sup\left\{\left(\sum_{k=1}^n |f\circ J_E(x_k)|^p\right)^{\frac{1}{p}} : (x_k)_k \subset E,\ \sup_{x^*\in B_{E^*}}\left(\sum_{k=1}^n |x^*(x_k)|^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

This provides an alternative representation for the norm in $\mathrm{FBL}^{(p)}[E^*]$ for any dual Banach space E^*.

$\mathrm{FBL}^{(p)}[E^{**}]$ vs $\mathrm{FBL}^{(p)}[E]^{**}$

Theorem (GS-Tradacete)

The operator $\widehat{\phi_E^{**}} : \mathrm{FBL}^{(p)}[E^{**}] \to \mathrm{FBL}^{(p)}[E]^{**}$ is an isometric lattice embedding.

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In other words:

The free p-convex Banach lattice generated by the free dual over the Banach space E ($\mathrm{FBL}^{(p)}[E^{**}]$) embeds lattice isometrically into the free dual over the free p-convex Banach lattice generated by E ($\mathrm{FBL}^{(p)}[E]^{**}$).

Overview

1. Preliminaries

2. $\mathrm{FBL}^{(p)}[E^{**}]$ vs $\mathrm{FBL}^{(p)}[E]^{**}$

3. Free dual Banach lattices

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- Does such a free object exists for every Banach space?
- If so, can we find an explicit construction?

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$$\begin{array}{ccc} \mathrm{FBL}^{*(p)}[E] & \xrightarrow{\ \check{T}\ } & X^* \\ {\scriptstyle \iota_E}\uparrow & \nearrow{\scriptstyle T} & \\ E & & \end{array}$$

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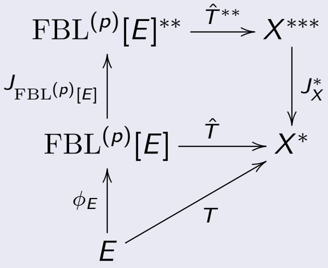

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In fact, $\mathrm{FBL}[E]^{**}$ fails to be the free dual Banach lattice over E as long as E contains a complemented copy of ℓ_1.

Theorem (GS-Tradacete)

Let E be a Banach space. The following are equivalent:

1. E does not contain a complemented copy of ℓ_1.
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1. E does not contain a complemented copy of ℓ_1.
2. The space $\mathrm{FBL}[E]^{**}$ satisfies the definition of $\mathrm{FBL}^*[E]$.

We do not know yet if $\mathrm{FBL}^*[E]$ exists when E contains a complemented copy of ℓ_1.

Thank you for your attention!

Main references

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