Positivity XI:

Dynamic programming principle and computable prices in financial market models with transaction costs.

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The problem of hedging (or super-hedging) a European claim

In discrete time $t = 0, \cdots, T$, let $(\Omega, (\mathcal{F}_t)_{t=0,\cdots,T}, P)$ be a discrete-time complete stochastic basis. Consider a $\mathcal{F}_T$-measurable random variable ξ_T we interpret as the payoff of some European option, i.e. a financial contract delivering the wealth ξ_T at time T.

The general problem is to solve the following : find a self-financing portfolio process $(V_t)_{t=0,\cdots,T}$ such that $V_T \geq \xi_T$ (or $V_T = \xi_T$ for an exact replication). We say that the initial value V_0 is a super-hedging price.

We are interested in the infimum of the super-hedging prices.

Portfolio processes and self-financing condition

Suppose that the financial market is composed of one bond of (discounted) price $S^1 = 1$ and $d - 1 \geq 1$ risky assets of prices $(S^i)_{i=2,\cdots,d}$.

A financial strategy $\theta \in \mathbf{R}^d$ is a stochastic process where θ^i_t is the number of assets number $i = 1, \cdots, d$ held by a portfolio manager.

Portfolio processes and self-financing condition : without transaction costs

The liquidation value of the financial strategy $\theta \in \mathbf{R}^d$ at time t is given by

$$L_t = L_t^\theta = \theta_t S_t = \sum_{i=1}^{d} \theta_t^i S_t^i.$$

The portfolio-process is said self-financing if, for all $t = 1, \cdots, T$,

$$\theta_{t-1} S_t = \theta_t S_t$$

or equivalently $\Delta L_t = L_t - L_{t-1}$ satisfies :

$$\Delta L_t = \theta_{t-1} \Delta S_t, t = 1, \cdots, T.$$

Portfolio processes and self-financing condition : with transaction costs

A portfolio process is expressed in physical units, i.e. $V_t = \theta_t$ and the liquidation value $L_t = L_t^{V_t}$ is not always simple to express. We consider the associated set-valued stochastic process $(G_t)_{t=0,\cdots,T}$ in $\mathbf{R}^d$ defined as

$$G_t := \{x \in \mathbf{R}^d : \; L_t(x) \geq 0\}.$$

If $\omega \mapsto L_t(\omega, x)$ is $\mathcal{F}_t$-measurable and $x \mapsto L_t(\omega, x)$ is upper semi-continuous, we may show that $G_t(\omega)$ is closed a.s.(ω) and measurable, where the measurability is understood in the graph sense :

$$\mathrm{graph}(G_t) := \{(\omega, x) : \; x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d), \; t = 1, \cdots, T.$$

Portfolio processes and self-financing condition : with transaction costs

Moreover, with $e_1 = (1, 0 \cdots, 0) \in \mathbf{R}^d$, we have

$$L_t(z) := \sup \{\alpha \in \mathbf{R} : z - \alpha e_1 \in G_t\} = \max \{\alpha \in \mathbf{R} : z - \alpha e_1 \in G_t\},$$

i.e. $L_t(z)$ is the largest amount of cash α we may obtain when we change $z = (z - \alpha e_1) + \alpha e_1$ into αe_1.

Similarly, if we define $C_t(z) = -L_t(-z)$, we obtain that

$$C_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in G_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in G_t\}, \quad z \in$$

i.e. $C_t(z)$ is the smallest cost α expressed in cash we need to buy the financial position $z \in \mathbf{R}^d$. Indeed, we write $\alpha e_1 = z + (\alpha e_1 - z)$.

Portfolio processes and self-financing condition : with transaction costs

Naturally, $C_t(z) = C_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable S_t of $\mathbf{R}_+^m$, $m \geq d$, and on the quantities $z \in \mathbf{R}^d$ to be traded.

In the following, we suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.

Portfolio processes and self-financing condition : with transaction costs

The self-financing condition is :

$$\Delta V_t \in -G_t,\ t = 0, \cdots, T, \quad (\Delta V_t = V_t - V_{t-1})$$

i.e. $V_{t-1} = V_t + (-\Delta V_t)$ is such that $C_t(\Delta V_t) \leq 0$ so that we may cancel the position $-\Delta V_t$ and change V_{t-1} into V_t for free.

We have random preorders $V_{t-1} \geq_{G_t} V_t$ defined by G_t, e.g. when G_t is a random convex closed cone.

Portfolio processes and self-financing condition : with transaction costs-Example

Suppose that the financial market is defined by an order book. In that case, we define S_t, at any time t, as

$$S_t = ((S_t^{b,i,j}, S_t^{a,i,j}), (N_t^{b,i,j}, N_t^{a,i,j}))_{i=1,\cdots,d, j=1,\cdots,k},$$

where k is the order book's depth and, for each $i = 1, \cdots, d$, $S_t^{b,i,j}, S_t^{a,i,j}$ are the bid and ask prices for asset i in the j-th line of the order book and $(N_t^{b,i,j}, N_t^{a,i,j}) \in (0, \infty]^2$ are the available quantities for these bid and ask prices. We suppose that $N_t^{b,i,k} = N_t^{a,i,k} = +\infty$ so that the market is liquid. By definition of the order book, we have $S_t^{b,i,1} > S_t^{b,i,2} > \cdots > S_t^{b,i,k}$ and $S_t^{a,i,1} < S_t^{a,i,2} < \cdots < S_t^{a,i,k}$.

Portfolio processes and self-financing condition : with transaction costs-Example

We then define the cost function as

$$C_t(x) = x^1 + \sum_{i=2}^{d} C_t^i(x^i), \quad x = (x^1, \cdots, x^d) \in \mathbf{R}^d.$$

With the convention $\sum_{r=1}^{j} = 0$ if $j = 0$, we consider the cumulated quantities $Q_t^{a,i,j} := \sum_{r=1}^{j} N_t^{a,i,r}$, $j = 0, \cdots, k$, the same for $Q_t^{b,i,j}$.

Portfolio processes and self-financing condition : with transaction costs-Example

We have :

$$
\begin{aligned}
C_t^i(y) &= \sum_{r=1}^{j} N_t^{a,i,r} S_t^{a,i,r} + (y - Q_t^{a,i,j}) S_t^{a,i,j+1}, \quad \text{if } Q_t^{a,i,j} < y \leq Q_t^{a,i,j+1}, \\
C_t^i(y) &= -\sum_{r=1}^{j} N_t^{b,i,r} S_t^{b,i,r} + (y + Q_t^{b,i,j}) S_t^{b,i,j+1}, \quad \text{if } -Q_t^{b,i,j+1} < y \leq -Q_t^{b,i}
\end{aligned}
$$

Note that the first expression of $C_t^i(z)$ above corresponds to the case where we buy $y > 0$ units of asset i. The second expression is $C_t^i(y) = -L_t^i(-y)$ when $y < 0$ so that $-C_t^i(y)$ is the liquidation value of the position $-y$, i.e. by selling the quantity $-y > 0$ at the bid prices.

The super-hedging problem without transaction costs

To solve the super-hedging problem on $[0, T]$, we suppose the no-arbitrage condition

NA : if $L = L^\theta$ is self-financing and satisfies $L_0 = 0$ and $L_T \geq 0$ a.s., then $L_T = 0$.

Theorem (Dalang–Morton–Willinger)

With $S^1 = 1$, NA is equivalent to the existence of a risk-neutral probability measure $Q \sim P$ such that S is a Q-martingale.

Let $\mathcal{M}(P)$ be the set of all risk-neutral probability measures $Q \sim P$ with $dQ/dP \in L^\infty$.

The super-hedging problem without transaction costs

Theorem

Suppose that NA holds and let ξ_T be a payoff integrable under P . There exists a minimal super-hedging price for ξ_T given by

$$\sup_{Q\in\mathcal{M}(P)} E_Q(\xi_T).$$

This is dificult to compute for many reasons : identifying the set $\mathcal{M}(P)$ and computing the supremum is not trivial !

The super-hedging problem with proportional transaction costs

Some no-arbitrage conditions are introduced for physical self-financing portfolio processes in the spirit of NA, e.g. the robust NA^r condition, see the Kabanov model, and we have :

> NA^r holds if and only if there exists strictly consistent price systems (SCPS), i.e. martingales Z of $\mathbf{R}^d$ such that, for all $t = 0, \cdots, d$, $Z_t \in G_t^* = \{y \in \mathbf{R}^d : \ xy \geq 0, \forall x \in G_t\}$.

Then, there exists a minimal price in cash for the European claim $\xi_T \in \mathbf{R}^d$ given by

$$\sup_{Z \in SCPS,\, Z_0 e_1 = 1} EZ_T \xi_T$$

The super-hedging problem for general transaction costs

In practice, the transaction costs are not necessary linear, see the case of order books or fixed costs.

The model is not linear so that we cannot expect dual elements characterizing a no-arbitrage condition that allow us to dually characterize the super-hedging prices.

Moreover, the no-arbitrage conditions we can imagine seem to be rather artificial, see the case of the Kabanov model where several distinct no-arbitrage exist and are difficult to compare.

The one period problem in general between time $T-1$ and T

As we suppose that $L_t(x) = x^1 + L_t((0, x^{(2)}))$ where $x^{(2)} = (x^2, \cdots, x^d) \in \mathbf{R}^d$, we have for some terminal wealth ξ_T such that $V_T = \xi$: $V_{T-1} \geq_{G_T} V_T \geq \xi_T$ is equivalent to

$$\begin{aligned}
L_T(V_{T-1} - \xi) \geq 0 \quad & \Longleftrightarrow V^1_{T-1} \geq \xi^1 - L_T((0, V^{(2)}_{T-1} - \xi^{(2)})), \\
& \Longleftrightarrow V^1_{T-1} \geq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 - L_T((0, V^{(2)}_{T-1} - \xi^{(2)}))\right), \\
& \Longleftrightarrow V^1_{T-1} \geq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + C_T((0, \xi^{(2)} - V^{(2)}_{T-1}))\right), \\
& \Longleftrightarrow V^1_{T-1} \geq F^{\xi}_{T-1}(V^{(2)}_{T-1}),
\end{aligned}$$

where

$$F^{\xi}_{T-1}(y) \quad := \quad \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + C_T((0, \xi^{(2)} - y))\right). \qquad (0.1)$$

The multi-step hedging problem

We denote by $\mathcal{P}_{T-1}(\xi)$ the set of all initial portfolio process values V_{T-1} at time $T-1$ that replicates $\xi = \xi_T$ at the terminal date T.

The infimum replicating cost is then defined as :

$$c_{T-1}(\xi) := \mathrm{ess\,inf}_{\mathcal{F}_{T-1}} \{C_{T-1}(V_{T-1}),\ V_{T-1} \in \mathcal{P}_{T-1}(\xi)\}.$$

The multi-step hedging problem

For $0 \leq t \leq T$ and $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we define $\gamma_t^\xi(V_{t-1})$ as :

$$\gamma_t^\xi(V_{t-1}) := \underset{V^{(2)} \in \Pi_t^T(V_{t-1},\xi)}{\operatorname{ess\,inf}_{\mathcal{F}_t}} \operatorname{ess\,sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t}^{T} C_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

Note that $\gamma_t^\xi(V_{t-1})$ is the infimum cost (cash) to replicate the payoff ξ when starting from the initial risky position $(0, V_{t-1}^{(2)})$ at time t.

We observe that $c_0(\xi) = \gamma_0^\xi(0)$ is the infimum cost to super-hedge ξ at time 0.

The multi-step hedging problem

Theorem (Dynamic Programming Principle)

For any $0 \leq t \leq T-1$ *and* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, *we have*

$$\gamma_t^{\xi}(V_{t-1}) = \operatorname*{ess\,inf}_{\mathcal{F}_t}{}_{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \operatorname{ess\,sup}_{\mathcal{F}_t} \left(C_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^{\xi}(V_t) \right). \tag{0.2}$$

The natural question is : how to compute the essential supremum and infimum above ?

The tools we use

Proposition

Let $h : \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^k)$-measurable function which is l.s.c. in x. Then, for all $X \in L^0(\mathbf{R}^k, \mathcal{F}_T)$,

$$\operatorname{ess\,sup}_{\mathcal{F}_t} h(X) = \sup_{x \in supp_{\mathcal{F}_t} X} h(x) \quad a.s.$$

Lemma

For any $\mathcal{F}$ normal integrand $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ and any non-empty $\mathcal{F}$-measurable closed set A, we have :

$$\operatorname{essinf}_{\mathcal{F}} \left\{ f(a),\ a \in L^0(A, \mathcal{F}) \right\} = \inf_{a \in A} f(a) \ a.s.$$

Pricing under SAIP

Theorem

Assume some technical conditions... Then,
$\gamma_t^\xi(V_{t-1}) = \gamma_t^\xi(S_t, V_{t-1})$ *is a* $\mathcal{F}_t$*-normal integrand of* S_t *and* V_{t-1} *and we have*

$$\gamma_t^\xi(S_t, V_{t-1}) = \inf_{y \in \mathbf{R}} \left(C_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^\xi(s, y) \right),$$

where $\phi_t(S_t) = \mathrm{supp}_{\mathcal{F}_t} S_{t+1}$.

Also, the infimum hedging cost of ξ *at any time* t *is reached, i.e.* $\gamma_t^\xi(V_{t-1})$ *is a mimimal cost.*

Conclusion

In discrete-time, it is possible to compute backwardly the infimum super-hedging prices in a large variety of models with transaction costs even if they are not linear.

To do so, it suffices to find some conditions under which the essential supremum and infimum of the dynamic programming principle are actually ω pointwise supremum and infimum respectively.

A weak no-arbitrage condition must ensure the finiteness of the infimum price to be computed. Without dual elements, we need to know the conditional supports of the process S that are a priori estimated directly from historical observations. A simple typical case is to suppose that $\mathrm{supp}_{\mathcal{F}_t} S^j_{t+1} = [k_t^{d,j} S^j_t, k_t^{u,j} S^j_t]$ as it is easy to calibrate $k_t^{d,j}$ and $k_t^{u,j}$ as minimum and maximum of the ratios S^j_{t+1}/S^j_t.

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Thank you for your attention !

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