The spectrum and essential spectra of some weighted composition operators on uniform algebras.

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Introduction

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- Let U be an isometric endomorphism of A. Then U is a composition operator,
$$(Uf)(m) = f(\varphi(m)), \ f \in A, \ a \in \mathfrak{M}_A,$$
- where φ is a continuous map of $\mathfrak{M}_A$ into itself such that $\varphi(\partial A) = \partial A$.

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- Let us agree about the following notations. Let X be a Banach space and $T : X \to X$ be a continuous linear operator. We denote the spectrum of T as $\sigma(T, X)$ and $\sigma_{a.p.}(T)$ means the approximate point spectrum of T, i.e.,
- $$\sigma_{a.p.}(T, X) = \{\lambda \in \mathbb{C} : \exists x_n \in X, \|x_n\| = 1, \, Tx_n - \lambda x_n \underset{n\to\infty}{\to} 0\}.$$

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- The upper semi-Fredholm spectrum of T, $\sigma_{usf}(T, X)$ is defined as

$$\sigma_{usf}(T, X) = \{\lambda \in \mathbb{C} : \exists x_n \in X, \|x_n\| = 1, Tx_n - \lambda x_n \underset{n\to\infty}{\to} 0,$$

and the sequence x_n is singular, i.e. it does not contain a norm convergent subsequence$\}$.

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- Nevertheless, if $T = wU$, where U is a **non-invertible** isometric endomorphism, our knowledge of the properties of the essential spectra of T remains absolutely unsatisfactory.
- The goal of this talk is to present some results that somewhat fill the vacuum.

The results

Theorem 1

Let A be a unital uniform algebra, T_φ is an isometric endomorphism of A and $w \in A$. Let $T = wT_\varphi$. Assume that φ is an open map of ∂A onto itself. Then,

$$\sigma_{usf}(T, A) = \sigma_{a.p.}(T, A) = \sigma_{a.p.}(T, C(\partial A)).$$

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The results

Corollary 2

Assume the conditions of Theorem 1. Assume also that the set of all φ-periodic points is of first category in ∂A. Then the set $\sigma_{usf}(T)$ is rotation invariant.

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Definition 3

A unital uniform algebra A is called **analytic** if for any open subset V of ∂A and any $f \in A$ the following implication holds $f|V \equiv 0 \Rightarrow f = 0$.

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Corollary 4

Assume the conditions of Theorem 1. Assume additionally that the uniform algebra A is analytic and that $T_\varphi^n \neq I, n \in \mathbb{N}$. Then the set $\sigma_{usf}(T)$ is rotation invariant.

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- In the next theorem we will investigate the conditions that guarantee that the set $\{|\lambda| : \lambda \in \sigma(T)\}$ is connected. In other words, we are interested in the presence (or the absence) of circular holes in the spectrum $\sigma(T, A)$.

The results

Theorem 5

Let A be a unital uniform algebra, T_φ is an isometric endomorphism of A and $w \in A$. Let $T = wT_\varphi$. Assume that φ is an open map of ∂A onto itself. The following conditions are equivalent.

(1). There is a positive real number r such that $\sigma(T) = \sigma_1 \cup \sigma_2$ where $\sigma_i \neq \emptyset, i = 1, 2$, $\sigma_1 \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$, and $\sigma_2 \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$.

(2). There is a closed ideal J of A such that

- a. *$TJ \subset J$ and $\sigma(T, J) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$.*
- b. *The algebra $A|h(J)$, where $h(J) \subset \mathfrak{M}_A$, is the hull of the ideal J, is closed in $C(h(J))$.*
- c. *The algebras $A|h(J)$ and $A|E$, where $E = h(J) \cap \partial A$, are isometric and $A|E$ is closed in $C(E)$.*
- d. *Moreover, $Int_{\partial A} E \neq \emptyset$.*
- e. *The operator $T = wT_\varphi$ acts on $A|E$ and $\sigma(T, A|E) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$.*

The results

- In the case when T_φ is an automorphism the statement in the previous theorem become much simpler: $\mathfrak{M}_A = \mathfrak{M}_1 \cup \mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_1$ are disjoint φ-invariant closed subsets of $\mathfrak{M}_A$.

$$\sigma(T, C(\mathfrak{M}_1) \subset r\mathbb{D}, \sigma(T, C(\mathfrak{M}_2)) \subset \mathbb{C} \setminus r\mathbb{D}.$$

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$$\sigma(T, C(\mathfrak{M}_1) \subset r\mathbb{D}, \sigma(T, C(\mathfrak{M}_2)) \subset \mathbb{C} \setminus r\mathbb{D}.$$

- In the case of non-invertible isometric endomorphisms we have currently to put up with the rather ugly conditions in Theorem 5. Nevertheless, we see that in the conditions of this theorem the hull of the ideal J contains an open subset of ∂A, which immediately provides the next corollary.

The results

Corollary 6

Assume conditions of Theorem 5. Assume additionally that the algebra A is analytic. Then the set $\{|\lambda| : \lambda \in \sigma(T, A)\}$ is connected. If we also assume that $T_\varphi^n \neq I, n \in \mathbb{N}$, then $\sigma(T, A)$ is a connected rotation invariant subset of $\mathbb{C}$.

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Corollary 6

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- We complement Theorems 1 and 5 by the following simple corollary.

The results

Corollary 7

Let A be a unital uniform algebra, T_φ be an isometric non-invertible endomorphism of A and $w \in A$. Assume that $|w| > 0$ on ∂A. Then $\rho_{min}(T)\mathbb{U} \subset \sigma_r(T, A)$, where

$$\rho_{min}(T) = \min_{\mu \in M_\varphi} \exp \int \ln |w| d\mu,$$

M_φ is the set of all φ-invariant probability measures in $C(\partial A)'$, and $\sigma_r(T, A) = \sigma(T, A) \setminus \sigma_{a.p.}(T, A)$.

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- Thus, $\sigma_r(T, A)$ consists of all the points in $\sigma(T, A)$ such that the operator $\lambda I - T$ has the left inverse.

Examples

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- In this example A is the disk-algebra: the algebra of all functions analytic in the open unit disk $\mathbb{U}$ and continuous in the $\mathbb{D} = cl\mathbb{U}$. We consider A as a closed subalgebra of $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Any weighted automorphism of A is of the form $T = wT_\varphi$, where $w \in A$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation.

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- The spectrum of weighted automorphisms of the disk-algebra was first described by the late Herbert Kamowitz in 1978. Here we will provide some additional details.

Examples

- Case 1. φ is an elliptic Möbius transformation and there is an $n \in \mathbb{N}$ such that $\varphi^n(z) \equiv z, z \in \mathbb{D}$ (i.e. φ is conformally equivalent to a periodic rotation). We assume that n is the smallest positive integer such that $T_\varphi^n = I$. Then,

$$\sigma(wT_\varphi) = \{\lambda \in \mathbb{C} : \exists z \in \mathbb{D} \text{ such that } \lambda^n = w_n(z)\},$$
$$\sigma_{a.p.}(wT_\varphi) = \sigma_{usf}(wT_\varphi) = \{\lambda \in \mathbb{C} : \exists z \in \mathbb{T} \text{ such that } \lambda^n = w_n(z)\}.$$

Examples

- Case 2. φ is either an elliptic Möbius transformation such that $T_\varphi^n \neq I, n \in \mathbb{N}$ or a parabolic Möbius transformation. Let ζ be the unique fixed point of φ in $\mathbb{D}$.
  (a) If w is invertible in A then
  $$\sigma_{usf}(wT_\varphi) = \sigma(wT_\varphi) = w(\zeta)\mathbb{T}.$$
  (b) If w is not invertible in A but $|w| > 0$ on $\mathbb{T}$ then
  $$\sigma(wT_\varphi) = w(\zeta)\mathbb{D} \text{ and } \sigma_{usf}(wT_\varphi) = w(\zeta)\mathbb{T}.$$
  (c) If $w(\xi) = 0$ for some $\xi \in \mathbb{T}$ then
  $$\sigma(wT_\varphi) = \sigma_{usf}(wT_\varphi) = w(\zeta)\mathbb{D}.$$

Examples

- Case 3. φ is a hyperbolic Möbius transformation. Let $\zeta_1, \zeta_2 \in \mathbb{T}$ be the fixed points of φ. Assume that $|\varphi'(\zeta_1)| < 1$ and $|\varphi'(\zeta_2)| > 1$. To avoid minor details we will assume that w is an invertible element of A.

  (a) If $|w(\zeta_1)| > |w(\zeta_2)|$ then

$$\sigma_{usf}(wT_\varphi) = \sigma(wT_\varphi) = \{\lambda \in \mathbb{C} : |w(\zeta_2| \leq |\lambda| \leq |w(\zeta_1)|\}.$$

  (b) If $|w(\zeta_2)| > |w(\zeta_1)|$ then

$$\sigma(wT_\varphi) = \{\lambda \in \mathbb{C} : |w(\zeta_2| \leq |\lambda| \leq |w(\zeta_1)|\},$$
$$\sigma_{usf}(wT_\varphi) = w(\zeta_1)\mathbb{T} \cup w(\zeta_2)\mathbb{T}.$$

  (c) If $|w(\zeta_2)| = |w(\zeta_1)|$ then

$$\sigma_{usf}(wT_\varphi) = \sigma(wT_\varphi) = w(\zeta_1)\mathbb{T}.$$

Examples

- **Weighted endomorphisms of the disk-algebra**.

  Any isometric endomorphism of the disk-algebra A is of the form

$$(T_\varphi f)(z) = f(B(z)), f \in A, z \in \mathbb{D},$$

where B is a finite Blaschke product. We assume that B has at least two factors, i.e., it is not a Möbius transformation.

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  where B is a finite Blaschke product. We assume that B has at least two factors, i.e., it is not a Möbius transformation.
- Let $w \in A$ and $T = wT_\varphi$. Theorems 1 and 5 guarantee that $\sigma(T)$ is a disk (or the singleton $\{0\}$) and that the sets $\sigma_{a.p.}(T) = \sigma_{usf}(T)$ are rotation invariant. The complicated dynamics of finite Blaschke products on the unit circle currently prevents us from obtaining more information about the spectrum and the essential spectra of T.

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- Even the computation of the spectral radius $\rho(T)$ becomes nontrivial (unless B is a Möbius transformation). The well-known formula

$$\rho(T) = \max_{\mu \in M_\varphi} \exp \int \ln |w| d\mu,$$

where M_B is the set of all B-invariant probability measures in $C(\mathbb{T})'$, becomes practically useless, because the set M_B is very large and does not allow a simple description.

Examples

- Let us consider for example the following two seemingly simple operators.

$$(T_1 f)(z) = \left(\frac{1+z}{2}\right) f(z^2),$$
$$(T_2 f)(z) = \left(\frac{1-z}{2}\right) f(z^2),$$
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\end{aligned}
$$

- It is trivial that $\rho(T_1) = 1$, and some elementary calculations show that $\rho(T_2) = \frac{1}{2}$. Thus, $\sigma(T_1) = \mathbb{D}$ and $\sigma(T_2) = \frac{1}{2}\mathbb{D}$. But, beyond the fact that the sets $\sigma_{a.p.}(T_i), i = 1, 2$, are rotation invariant and the trivial inclusions $\{0\} \cup \mathbb{T} \subset \sigma_{a.p.}(T_1)$ and $\{0\} \cup \frac{1}{2}\mathbb{T} \subset \sigma_{a.p.}(T_2)$ hold, we cannot say anything else about the sets $\sigma_{a.p.}(T_i), i = 1, 2$.

Examples

- I would like to end this talk by discussing one more simple example. Let A be the disk-algebra and

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- It is immediate to see that $\rho(T) = 3$ and $\rho_{min}(T) = 1$. Thus, $\sigma(T) = 3\mathbb{D}$ and by Corollary 7 we have $\mathbb{U} \subset \sigma_r(T)$.

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- It is immediate to see that $\rho(T) = 3$ and $\rho_{min}(T) = 1.$ Thus, $\sigma(T) = 3\mathbb{D}$ and by Corollary 7 we have $\mathbb{U} \subset \sigma_r(T)$.
- We do not know what happens inside the annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq 3\}$.