A localization principle in pre-Riesz spaces

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Pre-Riesz spaces

Definition/Theorem (van Haandel, 1993).

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Moreover, all vector lattices Y as in (iii) are isomorphic.

We call a pair (E, Φ) as in (ii) a *vector lattice cover of* X and as in (iii) the *Riesz completion of* X and denote it by (X^ρ, Φ).

Riesz* homomorphisms

Definition (van Haandel, 1993).

Let X, Y be povs. A linear map $T\colon X \to Y$ is called a *Riesz* homomorphism* if, for every non-empty finite subset F of X, one has

$$T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}}.$$

Riesz* homomorphism

Theorem (van Haandel, 1993).

Let X and Y be pre-Riesz spaces with Riesz completions (X^ρ, Φ_X) and (Y^ρ, Φ_Y), respectively. Let $T\colon X \to Y$ be a linear map. The following statements are equivalent:

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(i) T is a Riesz* homomorphism.

(ii) There exists a Riesz homomorphism $T^\rho\colon X^\rho \to Y^\rho$ satisfying $T^\rho \circ \Phi_X = \Phi_Y \circ T$.

Moreover, if (i) is satisfied, then the Riesz homomorphism T^ρ in (ii) is unique.

Riesz* homomorphisms

Proposition (Kalauch, van Gaans, 2019).
Let X, Y be pre-Riesz spaces and $T\colon X \to Y$ a Riesz* homomorphism. Then T is positive and disjointness preserving.

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"Proof"

$$\begin{array}{ccc} X^{\rho} & \xrightarrow{\;T^{\rho}\;} & Y^{\rho} \\ {\scriptstyle \Phi_X}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_Y} \\ X & \xrightarrow{\;T\;} & Y \end{array}$$

The converse is not true in general!

Riesz* homomorphisms on spaces of continuous functions

Theorem (van Imhoff, 2018).

Let P and Q be nonempty compact Hausdorff spaces and let X and Y be order dense subspaces of $\mathrm{C}(P)$ and $\mathrm{C}(Q)$, respectively. Let $T\colon X \to Y$ be linear. Then, under some mild conditions on X, the following statements are equivalent:

(i) T is a Riesz* homomorphism

(ii) There exist $w \in \mathrm{C}(Q)$, $w \geq 0$, and $\alpha\colon Q \to P$ continuous on $\{q \in Q;\, w(q) > 0\}$ such that

$$T(x)(q) = w(q)x(\alpha(q)) \quad (x \in X).$$

Order unit spaces

Definition.

Let X be a povs.

(a) An element $u \in X$ is called *order unit* if, for every $x \in X$, there is $\lambda \in (0, \infty)$ such that $\pm x \leq \lambda u$.

(b) If X is, in addition, Archimedean, then we can define a norm $\|x\|_u := \inf\{\lambda \in (0, \infty); -\lambda u \leq x \leq \lambda u\}$ $(x \in X)$ on X.

(c) If X is an Archimedean povs with order unit, then we call X an order unit space.

Note: Every order unit space is pre-Riesz.

Functional representation (Kadison, 1951)

Let X be an order unit space with order unit u. Define the weakly-$*$ compact convex set

$$\Sigma := \{\varphi \in X'; \varphi \text{ positive }, \varphi(u) = 1\}$$

and define Λ as the set of extreme points of Σ. The weak-$*$ closure $\overline{\Lambda}$ of Λ is a compact Hausdorff space (with the weak-$*$ topology) and the map

$$\Phi \colon X \to \mathrm{C}(\overline{\Lambda}), \quad x \mapsto (\varphi \mapsto \varphi(x)),$$

is linear and bipositive.

(Kalauch, Lemmens, van Gaans, 2014). $\Phi[X]$ is order dense in $\mathrm{C}(\overline{\Lambda})$.

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Definition.
Let X be a povs and $I \subseteq X$. We call I an *ideal* in X if I is a linear subspace and if

$$\forall x \in X, i \in I \colon \{x, -x\}^u \supseteq \{i, -i\}^u \text{ implies } x \in I.$$

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Note: Every directed and full subspace I of X is an ideal.
But: In contrast to vector lattices, there exist non-directed ideals!

Localization on principal ideals

Let X, Y be povs. For $x \in X$, $x > 0$, we denote by I_x the principal ideal generated by x, i.e.,

$$I_x := \{v \in X;\ \exists \lambda \in \mathbb{R}\colon\ \pm v \leq \lambda x\}.$$

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Note that x is an order unit in I_x.

$$T\colon X \to Y \text{ is Riesz* homomorphism}$$

$$\Big\downarrow \quad (x \in X, x > 0, y \geq T(x))$$

$$T|_{I_x}\colon I_x \to I_y \text{ Riesz* homomorphism?}$$

Pre-Riesz* subspaces

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Definition.
Let X be a povs and $U \subseteq X$ a subspace. Then we call U a *pre-Riesz* subspace* if the embedding $j_U \colon U \to X, u \mapsto u$, is a Riesz* homomorphism.

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Remark.

Let X, Y be povs and $T \colon X \to Y$ a Riesz* homomorphism. If $U \subseteq X$ is a pre-Riesz* subspace, then $T|_U \colon U \to Y$ is still a Riesz* homomorphism (as $T|_U = T \circ j_U$). This is not true for general subspaces!

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Definition.

A pre-Riesz space X with Riesz completion (X^ρ, Φ) is called *pervasive* if, for every $z \in X^\rho$, $z > 0$, there exists $x \in X$ such that $0 < \Phi(x) \leq z$.

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Theorem (S. 2023).

Let X be a povs, and I a directed ideal of X.

(a) If X has the RDP, or

(b) if X is an Archimedean pervasive pre-Riesz space,

then I is a pre-Riesz* subspace of X

Localization on principal ideals

We say that a povs X is *localizable* if the set

$$P_X := \{x \in X;\ x > 0,\ I_x \text{ is a pre-Riesz* subspace}\}$$

is majorizing in X.

Localizing on principal ideals

$T\colon X \to Y$ is Riesz* homomorphism

$$\Big\downarrow \quad (x \in P_X, y \geq T(x))$$

$T|_{I_x}\colon I_x \to I_y$ is a Riesz* homomorphism!

Localizing on principal ideals

$$T\colon X \to Y \text{ is Riesz* homomorphism}$$

$$\Big\downarrow \quad (x \in P_X, y \geq T(x))$$

$$T|_{I_x}\colon I_x \to I_y \text{ is a Riesz* homomorphism!}$$

Theorem (S., 2023).

If X is a localizable pre-Riesz space and for all $x \in P_X, y \geq T(x)$, the restriction $T|_{I_x}\colon I_x \to I_y$ is a Riesz* homomorphism, then T is a Riesz* homomorphism.

Localizing on principal ideals

Examples.

- Every order unit space with o.u. u is localizable as $u \in P_X$.
- Every Archimedean pervasive pre-Riesz space is localizable and we have $P_X = X_+ \setminus \{0\}$.
- Every povs space with RDP is localizable and we have $P_X = X_+ \setminus \{0\}$.

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- Every order unit space with o.u. u is localizable as $u \in P_X$.
- Every Archimedean pervasive pre-Riesz space is localizable and we have $P_X = X_+ \setminus \{0\}$.
- Every povs space with RDP is localizable and we have $P_X = X_+ \setminus \{0\}$.

Question.

Is every (Archimedean) pre-Riesz space localizable?

Localizing on principal ideals

Proposition, S. (2023)

Let X, Y be Archimedean pre-Riesz spaces. Then X and Y are localizable if and only if $X \times Y$ is localizable.

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Best idea so far

Best candidate right now for a pre-Riesz space, which might not be localizable, is the **space of self-adjoint compact operators on a Hilbert space**.

References I

[1] A. Kalauch, B. Lemmens, and O. van Gaans. Riesz completions, functional representations, and anti-lattices. Positivity **18**(1), 201-218 (2014).

[2] A. Kalauch and O. van Gaans. Pre-Riesz spaces. De Gruyter Expositions in Mathematics. De Gruyter, Berlin (2019).

[3] J. Stennder. A localization principle in pre-Riesz spaces. Preprint (2023).

[4] M. van Haandel. Completions in Riesz Space Theory. PhD thesis, University of Nijmegen (1993).

[5] H. van Imhoff. Riesz* homomorphisms on pre-Riesz spaces consisting of continuous functions. Positivity **22**(2), 425-447 (2018).

Thank you a lot for your attention.