Truncated normed Riesz space.

A representation of quasi-unitary Banach lattices.

HAMZA HAFSI

University of Tunis

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Plan

- Truncated Riesz spaces has been defined by Fremlin (1974) as Riesz subspaces of $\mathbb{R}^X$ satisfying **Stone's condition**, i.e., containing with any non-negative function f its meet $1 \wedge f$ with the constant function 1.

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- Quite recently, Ball (2014) provided an appropriate axiomatization of truncated Riesz spaces.

Introduction :

Definition

A **truncated Riesz space** we shall mean a Riesz space E along with a *truncation*, that is, a nonzero map $x \to x^*$ from the positive cone E^+ into itself such that

$$x \wedge y^* \leq x^* \leq x, \text{ for all } x, y \in E^+.$$

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Lemma

A nonzero map $x \to x^*$ is a truncation on E if and only if

$$x^* \wedge y = x \wedge y^*, \text{ for all } x, y \in E^+.$$

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- Boulabiar and El Adeb proved that if E is a truncated Riesz space, then the direct sum $E \oplus \mathbb{R}$ can be equipped with a non-standard structure of a Riesz space such that E is a Riesz subspace of $E \oplus \mathbb{R}$ and the equality

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- $E \oplus \mathbb{R}$ is a universal object.

Introduction :

Problem

How does the unitization $E \oplus \mathbb{R}$ *behave when the given truncated Riesz space* E *is also a normed Riesz space ?*

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- Accordingly, a necessary condition for $E \oplus \mathbb{R}$ to be equipped with a lattice norm that extends $\|.\|$ is that the truncation $x \to x^*$ must be norm-bounded, i.e., there exists $M \in (0, \infty)$ such that

$$\|x^*\| \leq M, \quad \text{for all } x \in E^+.$$

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Definition (Boulabiar,Hafsi. (2020))

An unitization norm $\|\cdot\|_u$ on $E \oplus \mathbb{R}$ is a lattice norm that extends the norm on E and satisfies $\|1\|_u = 1$.

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Extreme unitization norms :

The largest unitization norm

Theorem (Boulabiar,Hafsi. (2020))

Let E be a truncated normed Riesz space. The formula

$$\|x+\alpha\|_{\max} = \left\|\left(|x+\alpha| - |\alpha|\right)^{+}\right\| + |\alpha|, \quad \textit{for all } x \in E \textit{ and } \alpha \in \mathbb{R}$$

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Theorem (Boulabiar,Hafsi. (2020))

Let E be a truncated normed Riesz space with no truncation unit. Then the function that takes each $x+\alpha \in E \oplus \mathbb{R}$ to the positive real number

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In particular, E is closed in $(E \oplus \mathbb{R}, \|\|_{\max})$

(ii) *If E is dense in $(E \oplus \mathbb{R}, \|.\|_{\min})$ then E has no truncation unit and $\|\cdot\|_u = \|\cdot\|_{\min}$..*

Arbitrary unitization norms

Example

Let E be the set of continuous real-valued functions on $[-1,1]$ vsatisfying $f(1)=0$ equipped with the lattice norm

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Let E be truncated normed Riesz space and suppose that $E \oplus \mathbb{R}$ is equipped with any unitization norm.
Then, E is a Banach lattice if and only if $E \oplus \mathbb{R}$ is a Banach lattice.

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Representation of quasi-unitary Banach lattices

Definition (Boulabiar,Hafsi. (2020))

A Banach lattice E is said to be **quasi-unitary** if there exists a truncation map $x \to x^*$ on E satisfying :

$$\overline{B}_E(0,1) = E^* = \left\{x \in E : |x|^* = |x|\right\}$$

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Example

The function $f \to f^*$ defined on $C_0(X)$ by

$$f^* = \mathbf{1} \wedge f, \text{ for all } 0 \leq f \in C_0(X).$$

makes $C_0(X)$ a quasi-unitary Banach lattice.

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Corollary

Let X be a locally compact space. Then, $C_0(X) \oplus \mathbb{R}$ is isometrically isomorphic to $C(\omega X)$.

Thank you for your attention.