The order center and the algebraic center of JB-algebras

joint work with Anke Kalauch and Mark Roelands

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Introduction to pre-Riesz spaces

Let V be a vector space over $\mathbb{R}$. $C \subseteq V$ is a cone if $C + C \subseteq C$, $\lambda C \subseteq C$ for all $\lambda \in [0, \infty)$, and $C \cap -C = \{0\}$.
Partial order $\leq$ on V given by $x \leq y \iff y - x \in C$.
(V, C) is a partially ordered vector space.

A partially ordered vector space (V, C) is a Riesz space or vector lattice if for all $x, y \in V$ the set $\{x, y\}$ has a supremum.

Why partially ordered vector spaces instead of Riesz spaces?

- subspaces of Riesz spaces
- spaces of operators between Riesz spaces

are partially ordered vector spaces but not always Riesz spaces

A partially ordered vector space (V, C) is

- directed if $C - C = V$ and
- Archimedean if $nx \leq y$ for all $n \in \mathbb{N}$ implies $x \leq 0$.

Introduction to pre-Riesz spaces

Idea: View partially ordered vector spaces as subspaces of Riesz spaces and use theory of Riesz spaces.

W. Luxemburg (1986): for every partially ordered vector space V there exists a Riesz space Y and a bipositive linear map $i\colon V \to Y$.
bipositive: $i(x) \geq 0 \iff x \geq 0$

Let V partially ordered vector space, Y Riesz space, $V \subseteq Y$ linear subspace, $x, y \in V$.
Say x, y are disjoint in V if they are disjoint in Y.
Problem: does depend on Y.

V is order dense in Y if for every $y \in Y$

$$y = \inf\{v\colon\ v \in V,\ v \geq y\}$$ Buskes and van Rooij, 1993

If $x, y \in V$, V order dense in Y_1 and order dense in Y_2, then x and y disjoint in Y_1 if and only if disjoint in Y_2.

Introduction to pre-Riesz spaces

Approach:

- Write definition in terms of upper and lower bounds.

In a Riesz space:
x, y are disjoint $\iff |x-y| = |x+y| \iff$
$\{x-y, -x+y\}^{\mathrm{u}} = \{x+y, -x-y\}^{\mathrm{u}}$

- Show compatibility with order dense embedding.

If V order dense in Y, then x, y disjoint in V if and only if x, y disjoint in Y.

- Use theory of Riesz spaces.

For $A \subseteq V$, $A^{\mathrm{d}} := \{v \in V : \ v$ and a disjoint for all $a \in A\}$ is a linear subspace of V.

Introduction to pre-Riesz spaces

Which partially ordered vector spaces can be embedded order densely in Riesz spaces?

A partially ordered vector space (V, C) is a pre-Riesz space if for all $x, y, z \in V$ $\{x+z, y+z\}^{\mathrm{u}} \subseteq \{x, y\}^{\mathrm{u}}$ implies $z \geq 0$.

Theorem (van Haandel, 1993)

Let (V, C) be a partially ordered vector space. Then V is pre-Riesz if and only if there exists a vector lattice Y and a bipositive map $i\colon V \to Y$ such that $i[V]$ is order dense in Y.

(Y, i) is then called a vector lattice cover of V. There is a unique smallest vector lattice cover, called the Riesz completion of V.

Van Haandel (1993): (V, C) directed and Archimedean $\Longrightarrow$ (V, C) is pre-Riesz $\Longrightarrow$ (V, C) is directed.

order dense: $x = \inf\{d \in D\colon\ d \geq x\}$.

bipositive: $x \geq 0$ in $V \iff i(x) \geq 0$ in W.

Introduction to pre-Riesz spaces

(V, C, e) order unit space:

- $\exists e \in V$ which is an order unit if $\forall x \in V\ \exists \lambda \in \mathbb{R}$ such that $-\lambda e \leq x \leq \lambda e$.
- (V, C) is Archimedean;

$$\|x\|_e = \inf\{\lambda \in \mathbb{R}\colon\ -\lambda e \leq x \leq \lambda e\}$$ order unit norm.

Note that an order unit space is directed.

Hence order unit spaces are pre-Riesz spaces.

Introduction to pre-Riesz spaces

Let (V, C, e) be an order unit space. Vector lattice cover of V?

Functional representation of V:

$\Sigma = \{\varphi\colon V \to \mathbb{R}\colon \varphi \text{ positive linear}, \varphi(e) = 1\}$
$\Lambda =$ extreme points of Σ
$\overline{\Lambda} =$ weak* closure of Λ in Σ
$\Phi(x)(\varphi) = \varphi(x)$, $\varphi \in \overline{\Lambda}$, $x \in V$

Kalauch–Lemmens–vG (2014):
$(C(\overline{\Lambda}), \Phi)$ is a vector lattice cover of V, i.e.,
$\Phi\colon V \to C(\overline{\Lambda})$ is bipositive and $\Phi[V]$ is order dense in $C(\overline{\Lambda})$

More talks on pre-Riesz spaces: Anke Kalauch, Florian Boisen, Janko Stennder

Introduction to JB-algebras

See Alfsen–Shultz, *Geometry of state spaces of operator algebras*, 2003.

Let Ω be a compact Hausdorff space,
$C(\Omega) = \{x \colon \Omega \to \mathbb{R} \colon\ x \text{ is continuous}\}$. This is a vector space over $\mathbb{R}$

- with a product: $(xy)(\omega) = x(\omega)y(\omega)$, $s \in \Omega$
- and a partial order: $x \leq y \iff \forall \omega \in \Omega \colon x(s) \leq y(s)$.
- The constant function 1 is an order unit and an identity for the product.
- Note that $x \geq 0$ if and only if $\exists y$ such that $x = y^2$ (namely, $y = \sqrt{x}$), so $C(\Omega)^+ = \{x^2 \colon\ x \in C(\Omega)\}$.

$C(\Omega)$ is a commutative C^*-algebra.

Introduction to JB-algebras

Let H be a Hilbert space over $\mathbb{C}$,
$B(H) = \{x\colon H \to H\colon\ x \text{ is bounded linear}\}$. This is a vector space over $\mathbb{R}$

- with a product: composition
- and a partial order: x is positive if $\sigma(x) \subseteq [0, \infty)$,

$x \leq y \iff y - x \geq 0$.

- The identity operator is an identity for the product, but not an order unit. Actually, $B(H)^+$ does not even span $B(H)$ (over $\mathbb{R}$).

Introduction to JB-algebras

Rather consider $B(H)_{\mathrm{sa}} = \{x \in B(H) \colon\ x \text{ is self adjoint}\}$.
But $B(H)_{\mathrm{sa}}$ is not closed under composition:
$(xy)^* = y^*x^* = yx \neq xy$.
Consider the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$.
Note: $x \circ x = xx$, so we can unambiguously write x^2.

The Jordan product on $B(H)_{\mathrm{sa}}$ distributes over sums:

- $(x + y) \circ z = x \circ z + y \circ z$ and $x \circ (y + z) = x \circ y + x \circ z$
- is commutative: $x \circ y = y \circ x$

but not associative:

$$(x \circ y) \circ z = \frac{1}{2}\Big((x \circ y)z + z(x \circ y)\Big) = \frac{1}{2}\Big(\frac{1}{2}(xy + yx)z + \frac{1}{2}z(xy + yx)\Big)$$

and

$$x \circ (y \circ z) = \frac{1}{2}\Big(x(y \circ z) + (y \circ z)x\Big) = \frac{1}{2}\Big(\frac{1}{2}x(yz + zy) + \frac{1}{2}(yz + zy)x\Big),$$

- so $(x \circ y) \circ z \neq x \circ (y \circ z)$.

A very weak form of associativity still holds true:

- $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ Jordan identity.

Introduction to JB-algebras

$B(H)_{\mathrm{sa}} = \{x \in B(H) \colon x \text{ is self adjoint}\}$,
Jordan product $x \circ y = \frac{1}{2}(xy + yx)$
distributes over sums, is commutative, not associative, and

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \qquad \text{Jordan identity.}$$

The identity operator I is an identity for the Jordan product and I is an order unit:
$\forall x \in B(H)_{\mathrm{sa}}\ \exists \lambda \in \mathbb{R}$ such that $-\lambda I \leq x \leq \lambda I$.

Note that $B(H)^+_{\mathrm{sa}} = \{x^2 \colon x \in B(H)_{\mathrm{sa}}\}$.

Compatibility between the norm and the Jordan product:
$\|x \circ y\| \leq \|x\|\|y\|, \|x^2\| = \|x\|^2, \|x^2\| \leq \|x^2 + y^2\|$.

Introduction to JB-algebras

Two more notes on the Jordan product.

Recall the Jordan identity: $x \circ (y \circ x^2) = (x \circ y) \circ x^2$:

• powers of x: $x^2 = x \circ x = xx$, $x^3 = x \circ (x \circ x) = x \circ x^2 = x^2 \circ x$.
How about x^4? By Jordan identity,
$x \circ x^3 = x \circ (x \circ x^2) = (x \circ x) \circ x^2 = x^2 \circ x^2 = x^3 \circ x$,
so x^4 is unambiguously defined. Similar for x^n.

• Consider the left multiplication by $a \in A$, $T_a x = a \circ x$ for all $x \in A$.
$T_a T_b = T_b T_a$ if and only if $\forall x \in A$: $a \circ (b \circ x) = b \circ (a \circ x)$, or, equivalently, $(a \circ x) \circ b = a \circ (x \circ b)$.
a and b are then said to operator commute.

Introduction to JB-algebras

Definition

A Jordan algebra $(A, \circ)$ is a commutative, not necessarily associative algebra such that

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \text{ for all } x, y \in A.$$

A JB-algebra $(A, \circ)$ is a Jordan algebra over $\mathbb{R}$ with a norm $\|\cdot\|$ such that it is norm complete and

$$\|x \circ y\| \leq \|x\|\|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2 + y^2\|$$

for all $x, y \in A$.

JB-algebras as order unit spaces

Recall

Definition

A JB-algebra $(A, \circ)$ is a commutative, not necessarily associative algebra over $\mathbb{R}$ such that

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \text{ for all } x, y \in A.$$

endowed with a norm which makes it complete and

$$\|x \circ y\| \leq \|x\| \|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2 + y^2\|$$

for all $x, y \in A$.

We assume A has an algebraic identity e.

It is known that

- $C = \{x^2 \colon x \in A\}$ is a closed cone in A,
- e is an order unit in A,
- the norm of A equals the order unit norm.

So, with $x \leq y \iff y - x \in C$, A is an order unit space.

JB-algebras as order unit spaces

Example

- $C(\Omega)$, $x \circ y$ pointwise product, constant 1 function is the identity, maximum-norm, pointwise order. This is an associative JB-algebra.

- $B(H)_{\mathrm{sa}}$, Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, identity operator is the identity, operator norm, ordered by the cone of 'being positive definite'.

- Spin factor: $H \times \mathbb{R}$, where $(H, \langle \cdot, \cdot, \rangle)$ is a real Hilbert space, product
$$(x, \alpha) \circ (y, \beta) = (\beta x + \alpha y, \langle x, y \rangle + \alpha\beta),$$
$(0, 1)$ is the identity, norm $\|(x, \alpha)\| = \sqrt{\langle x, x \rangle} + |\alpha|$, ordered by the Lorentz cone $\{(x, \alpha)\colon \sqrt{\langle x, x \rangle} \leq \alpha\}$.

The following is known:

Theorem

Each associative unital JB-algebra is as JB-algebra isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω.

JB-algebras as order unit spaces

Let $(A, \circ)$ be a JB-algebra with identity e.

Question: How are the order structure and algebra structure related?

Algebraic center of A:

$$Z(A) = \{z \in A\colon\ \forall a \in A\colon\ T_z T_a = T_a T_z\},$$

where $T_a x = a \circ x$ (left multiplication by a).

Order center of A:

$$E(A) = \{T\colon A \to A \text{ linear}\colon\ \exists \alpha \in \mathbb{R} \text{ such that } -\alpha I \leq T \leq \alpha I\},$$

where I is the identity operator on A.

Theorem: $Z(A)$ and $E(A)$ are isomorphic as JB-algebras.

Order center

Let (V, C) be an Archimedean directed partially ordered vector space.

Order center of V:
$E(A) = \{T\colon A \to A \text{ linear}\colon\ \exists\alpha \in \mathbb{R} \text{ s.t. } -\alpha I \leq T \leq \alpha I\} \subseteq L^r(V)$.
is a partially ordered vector space with order unit I.

• Well-known: If $V = C(\Omega)$ then $T \in E(V)$ if and only if $\exists v \in V$ such that $T = M_v$, where

$$(M_v x)(\omega) = v(\omega)x(\omega),\ \omega \in \Omega,\ x \in V.$$ multiplication operator

• R.C. Buck (1961): V is isomorphic to a subspace of some $C(\Omega)$ and elements of $E(V)$ correspond to multiplication operators.

• If (V, C, e) is an order unit space and $\Phi\colon V \to \mathrm{C}(\overline{\Lambda})$ its functional representation, then for every $T \in E(V)$ the multiplication operator $M_{\Phi(Te)}$ maps $\Phi[V]$ into $\Phi[V]$ and $T = \Phi^{-1} \circ M_{\Phi(Te)} \circ \Phi$.

Order center

Let (V, C, e) be an order unit space.
$E(A) = \{T : A \to A \text{ linear} : \exists \alpha \in \mathbb{R} \text{ s.t. } -\alpha I \leq T \leq \alpha I\}$
$T \in E(V) \implies T = \Phi^{-1} \circ M_{\Phi(Te)} \circ \Phi.$

Proposition

- $\forall S, T \in E(V)$: $ST \in E(V)$
- $\forall S, T \in E(V)$: $ST = \Phi^{-1} \circ M_{\Phi(Se)\Phi(Te)} \circ \Phi = TS.$
- $T \in E(V) \implies T$ is continuous.
- On $E(V)$ the operator norm and the norm induced by the order unit I are equal.
- $E(V)$ is a closed subspace of the bounded linear operators on V.
- $\forall S, T \in E(V)$: $\|ST\| \leq \|S\|\|T\|$, $\|T^2\| = \|T\|^2$, and $\|T^2\| \leq \|S^2 + T^2\|$.

Order center

Corollary

- *$E(V)$ with composition is a commutative associative algebra.*
- *If (V, C, e) is norm complete, then $E(V)$ is an associative unital JB-algebra, hence isomorphic to $C(\Omega)$ as JB-algebras.*

Corollary

If (V, C, e) is norm complete, then $E(V)$ is a vector lattice.

Example

$V = C^1[0,1]$, pointwise order, $e = 1$. Order unit norm is $\| \cdot \|_\infty$, not complete. We have $E(V) = \{M_f : f \in V\}$, which is not a vector lattice.

Algebraic center and order center

Let $(A, \circ)$ be a JB-algebra with identity e.

Algebraic center of A:

$$Z(A) = \{z \in A \colon \forall a \in A \colon T_z T_a = T_a T_z\},$$

where $T_a x = a \circ x$ (left multiplication by a).

$Z(A)$ is a JB-subalgebra of A and it is associative, hence JB-algebra isomorphic to a $C(\Omega)$.

Algebraic center and order center

Theorem: Let $(A, \circ)$ be a JB-algebra with identity e.

- The algebraic center $Z(A)$ and the order center $E(A)$ are isomorphic as JB-algebras.
- $f(z) = T_z$ is a JB-algebra isomorphism from $Z(A)$ onto $E(A)$.

$T_z x = z \circ x$

$B(A) = \{T\colon A \to A\colon\; T$ linear and bounded$\}$ with the operator norm.

It is routine to show:

Lemma

$f\colon Z(A) \to B(A)$ is linear, multiplicative, injective, and $f(e) = I$ (with I the identity operator on A). Moreover, $\|T_z\| = \|z\|$ for all $z \in Z(A)$.

Remains to show: f maps into and onto $E(A)$.

Algebraic center and order center

Goal: show that $f\colon z \mapsto T_z$ maps $Z(A)$ onto $E(A)$.

Strategy: show that f is a bijection from $[0, e] \cap Z(A)$ onto $[0, I]$ by means of extreme points.

We need enough extreme points. Therefore, consider JBW-algebras first.

Definition: A JBW-algebra M is a JB-algebra which is the dual space of some Banach space M_*.

Algebraic center and order center

Let M be a JBW-algebra with identity e.
$p \in M$ is central projection if $p^2 = p$ and $p \in Z(M)$.

Lemma

- *The extreme points of $[0, e]$ are precisely the central projections in M.*
- *The extreme points of $[0, I]$ are of the form T_p for some central projection p in M with $p \in Z(M)$.*

Hence $z \mapsto T_z$ maps extreme points of $[0, e] \cap Z(M)$ onto extreme points of $[0, I]$.

- $[0, e] \cap Z(M)$ is convex and σ-weakly compact.
- $[0, I]$ is convex and compact for the σ-weak operator topology. (Choi and Kim, 2008)
- $z \mapsto T_z \colon M \to B(M)$ is continuous with respect to the σ-weak topology on $[0, e] \cap Z(M)$ and the σ-weak operator topology on $B(M)$.

Algebraic center and order center

Theorem

Let M be a JBW-algebra with identity e. Then

- $f : z \mapsto T_z : Z(M) \to E(M)$ *is a linear, multiplicative, isometric bijection.*
- *the map f is also a homeomorphism for the σ-weak topology of M on $Z(M)$ and the σ-weak operator topology of $B(M)$ on $E(M)$.*

The bidual A^{**} of a JB-algebra A is a JBW-algebra. Thus:

Theorem

Let A be a JB-algebra with identity e. Then

- $f : z \mapsto T_z : Z(A) \to E(A)$ *is an isometric isomorphism of JB-algebras.*
- *on $E(A)$ the operator norm and the order unit norm induced by I are equal*
- *The map f is also a homeomorphism for the weak topology of A on $Z(A)$ and the weak operator topology of $B(A)$ on $E(A)$.*

Summary

Let A be a JB-algebra, i.e., a Banach space with a product $\circ$ such that it is a commutative, not necessarily associative algebra over $\mathbb{R}$ with

$x \circ (y \circ x^2) = (x \circ y) \circ x^2$ for all $x, y \in A$ and

$\|x \circ y\| \leq \|x\|\|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2 + y^2\|.$

Algebraic center of A: $Z(A) = \{z \in A \colon \forall a \in A \colon T_z T_a = T_a T_z\}$, where $T_a x = a \circ x$ (left multiplication by a).

Order center: $E(A) = \{T \colon A \to A \colon \exists \alpha \in \mathbb{R} \text{ s.t. } -\alpha I \leq T \leq \alpha I\}$.

Theorem

If (V, C, e) is a norm complete order unit space, then $E(V)$ is a JB-algebra.

Theorem

The map $a \mapsto T_a \colon Z(A) \to E(A)$ is an isomorphism of JB-algebras,

Summary

Let A be a JB-algebra, i.e., a Banach space with a product $\circ$ such that it is a commutative, not necessarily associative algebra over $\mathbb{R}$ with

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \text{ for all } x, y \in A \text{ and}$$
$$\|x \circ y\| \leq \|x\|\|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2 + y^2\|.$$

Algebraic center of A: $Z(A) = \{z \in A\colon \forall a \in A\colon T_z T_a = T_a T_z\}$, where $T_a x = a \circ x$ (left multiplication by a).

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The map $a \mapsto T_a\colon Z(A) \to E(A)$ is an isomorphism of JB-algebras,

THANK YOU!

Algebraic center and order center

Goal: show that $f : z \mapsto T_z$ maps $Z(A)$ onto $E(A)$.

Strategy: show that f is a bijection from $[0, e] \cap Z(A)$ onto $[0, I]$ by means of extreme points.

Lemma

The extreme points of $[0, e]$ are precisely the projections in A.

$p \in A$ is a projection if $p^2 = p$.

We need enough extreme points. Therefore, consider JBW-algebras first.

Definition: A JBW-algebra M is a JB-algebra which is the dual space of some Banach space M_*.

Algebraic center and order center

A JBW-algebra M is a JB-algebra which is the dual space of some Banach space M_*.
Let M be a JBW-algebra with identity e.

$\varphi \in M^*$ is a normal state if it is a state (i.e., φ is positive and $\varphi(e) = 1$) and for every bounded increasing net $(x_i)_i$ with supremum x we have $\varphi(x_i) \to \varphi(x)$.

For $(x_i)_i$ and x in M we say that $x_i \to x$ σ-weakly if $\varphi(x_i) \to \varphi(x)$ for every normal state φ on M.

For bounded linear $(T_i)_i$ and T on M we say that $T_i \to T$ in the σ-weak operator topology if $\varphi(T_i x) \to \varphi(Tx)$ for all $x \in M$ and all normal states φ on M.

Algebraic center and order center

Let M be a JBW-algebra with identity e.

Lemma

- *The convex set $[0, e] \cap Z(M)$ in M is σ-weakly compact.*
- *The convex set $[0, I]$ in $B(M)$ is compact for the σ-weak operator topology. (Choi and Kim, 2008)*

Lemma

The map $z \mapsto T_z : M \to B(M)$ is continuous with respect to the σ-weak topology on $[0, e] \cap Z(M)$ and the σ-weak operator topology on $B(M)$.

Lemma

- *The extreme points of $[0, e]$ are precisely the projections in M.*
- *The extreme points of $[0, I]$ are of the form T_p for some projection p in M with $p \in Z(M)$.*

Algebraic center and order center

Claim: $f : z \mapsto T_z$ maps $[0, e] \cap Z(M)$ into $[0, I]$:
Indeed, let $z \in Z(M)$. Then

- z extreme point of $[0, e] \cap Z(M)$

$\Longrightarrow$ z is projection in M and $z \in Z(M)$

$\Longrightarrow$ $T_z \in [0, I]$

- z convex combination of extreme points of $[0, e] \cap Z(M)$

$\Longrightarrow$ $T_z \in [0, I]$

- As $[0, e] \cap Z(M)$ is compact and convex, by Krein-Milman,

$\forall z \in [0, e] \cap Z(M)$ we have $T_z \in [0, I]$.

Algebraic center and order center

Claim: $f : z \mapsto T_z$ maps $[0, e] \cap Z(M)$ onto $[0, I]$:
Indeed, let $T \in [0, I]$. Then

- T extreme point of $[0, I]$
$\implies \exists$ projection $p \in M$ with $p \in Z(M)$ such that $T_p = T$
$\implies T \in f[[0, e] \cap Z(M)]$.

- T convex combination of extreme points of $[0, I]$
$\longrightarrow T \in f[[0, e] \cap Z(M)]$.

- As $[0, I]$ is compact and convex, by Krein-Milman,
$\forall T \in [0, I]$ we have $T \in f[[0, e] \cap Z(M)]$.

Algebraic center and order center

Theorem

Let M be a JBW-algebra with identity e. Then

- $f : z \mapsto T_z : Z(M) \to E(M)$ *is a linear, multiplicative, isometric bijection.*
- *the map f is also a homeomorphism for the σ-weak topology of M on $Z(M)$ and the σ-weak operator topology of $B(M)$ on $E(M)$.*

The bidual A^{**} of a JB-algebra A is a JBW-algebra. Thus:

Theorem

Let A be a JB-algebra with identity e. Then

- $f : z \mapsto T_z : Z(A) \to E(A)$ *is an isometric isomorphism of JB-algebras.*
- *on $E(A)$ the operator norm and the order unit norm induced by I are equal*
- *The map f is also a homeomorphism for the weak topology of A on $Z(A)$ and the weak operator topology of $B(A)$ on $E(A)$.*