Truncated vector lattices: Something old and something new

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Some history

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Definition

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Theorem (Stone, 1948)

If the vector sublattice L of $\mathbb{R}^X$ satisfies the Stone condition, then for any σ-order continuous linear functional ψ on L, there exists a measure λ on X such that

$$\psi f = \int_X f d\lambda \quad \textit{for all } f \in L.$$

Definition (Fremlin, 1974)

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Theorem (Fremlin, 1974)

Let L be a vector lattice with a Fatou M-norm such that the supremum

$$\sup \left\{ [0, f] \cap \overline{B} (0, \alpha) \right\}$$

exists in L for all $f \in L^+$ and $\alpha \in (0, \infty)$. Then L is (lattice isomorphic to) a truncated vector sublattice of $\ell^\infty (X)$ for some X.

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Let A be an f-algebra with unit e. A subset S of A is said to have the **Stone condition** if $e \wedge f \in S$ for all $f \in S$.

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Then, $(i) \wedge (j) \Rightarrow (k)$ whenever i, j, k are pairwise different in $\{1, 2, 3\}$.

THE AXIOMATIZATION: A MILESTONE !

DEFINITION (BALL, 2014)

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LEMMA

A vector lattice L is truncated if and only if there exists a unary operation $*$ on L such that

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1. $0^* = 0$ *and* $f^* \wedge g = f \wedge g^*$ *for all* $f, g \in L$, *and*
2. $\{f \in L^+ : (nf)^* = nf \text{ for all } n \in \mathbb{N}\} = \{0\}$.

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EXAMPLE

If X is a locally compact Hausdorff space, then $C_0(X)$ is a truncated vector lattice with respect to its **canonical** truncation defined by $f^* = 1 \wedge f$ for all $f \in C_0(X)$. Moreover, $C_0(X)$ is unital if and only if X is compact.

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THEOREM (BALL, 2014)

For any weakly truncated Archimedean vector lattice L, there exists a locally compact Hausdorff space X such that L is (lattice isomorphic with) a vector lattice of functions in $C^\infty(X)$ and $f^* = 1 \wedge f$ for all $f \in L^+$.

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PROBLEM

Ball proved that any weakly truncated Archimedean vector lattice has a unitization in ZFC-set theory. The starting point was the question of whether or not the result holds in ZF-set theory (Zaanen Program).

Alexandroff unitization

Definition

Let L, M be two truncated vector lattices. A linear map $T : L \longrightarrow M$ is called a **truncation homomorphism** if

$$T(f^*) = (Tf)^* \quad \text{for all } f \in L.$$

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A bijective truncation homomorphism $T : L \rightarrow M$ is called a **truncation isomorphism**.

THEOREM

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DEFINITION

Let L, M be two unital truncated vector lattices with truncation units u, v respectively. A linear operator $T : L \longrightarrow M$ is said to be **unital** or **identity preserving** if $Tu = v$.

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Let L, M be two unital truncated vector lattices with truncation units u, v respectively. A linear operator $T : L \longrightarrow M$ is said to be **unital** or **identity preserving** if $Tu = v$.

LEMMA

A unital lattice homomorphism between two unital truncated vector lattices is a truncation homomorphism.

Theorem

Let L be a truncated vector lattice. There exists a unique (up to a unital lattice isomorphism that leaves L pointwise fixed) unitization αL of L such that, for every unital truncated vector lattice U, any truncation homomorphism $T : L \longrightarrow U$ extends uniquely to a unital lattice homomorphism $T^{\alpha} : \alpha L \longrightarrow U$:

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$$\begin{array}{ccccc} L & & \xrightarrow{T} & & U \\ & \iota \searrow & & \nearrow T^{\alpha} & \\ & & \alpha L & & \end{array}$$

COROLLARY

If the truncated vector lattice L is not unital then αL is the unique (up to a unital lattice isomorphism that leaves L pointwise fixed) unitization L^{*} of L such that, for every unital truncated vector lattice U, any one-to-one truncation homomorphism $T : L \longrightarrow U$ extends uniquely to a one-to-one unital lattice homomorphism $T^{*} : L^{*} \longrightarrow U$.

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Let L be a truncated vector lattice.

1. *The direct sum $L \oplus \mathbb{R}$ is a vector lattice whose positive cone is the union*

$$[L \oplus \mathbb{R}]^{+} = L^{+} \cup \left\{ f + r : r > 0 \text{ and } \left(\frac{1}{r} f^{-}\right)^{*} = \frac{1}{r} f^{-} \right\}.$$

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2. *$L \oplus \mathbb{R}$ is an Alexandroff unitization of L.*

Lattice norms on the Alexandroff unitization

Problem

Let L be a truncated vector lattice with a lattice norm . We want to know whether or not $\|.\|$ extends to a lattice norm $\|.\|_u$ on $\alpha L = L \oplus \mathbb{R}$?

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Fact

If so, the set of positive fixed points of the truncation must be norm-bounded.

Definition

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Let L be a normed truncated vector lattice whose norm is denoted by $\|.\|$. A lattice norm $\|.\|_u$ on $L \oplus \mathbb{R}$ is called a **unitization norm** if $\|1\|_u = 1$ and $\|f\|_u = \|f\|$ for all $f \in L$.

Theorem

Let L be a normed truncated vector lattice. The formula

$$\|f+r\|_{u,1} = \left\|\left(|f+r|-|r|\right)^{+}\right\| + |r| \quad \textit{for all } f \in L \textit{ and } r \in \mathbb{R}$$

defines the largest unitization norm on $L \oplus \mathbb{R}$.

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THEOREM

Let L be a normed truncated vector lattice. If L has no unit, the gauge function

$$\|f+r\|_{u,0} = \sup\left\{\|g\| : |g| \leq |f+r|\right\} \quad \text{for all } f \in L \text{ and } r \in \mathbb{R}$$

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2. *If L is dense in $L \oplus \mathbb{R}$ then L is not unital and $\|.\|_u = \|.\|_{u,0}$.*
3. *L is a Banach lattice if and only if $L \oplus \mathbb{R}$ is a Banach lattice.*

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2. *There exists a clopen set Y of X such that $f^{*} = 1_{Y} \wedge f$ for all $f \in L$.*

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1. *L is (lattice isomorphic with) an order dense vector sublattice of $C^{\infty}(X)$,*
2. *$f^{*} = 1 \wedge f$ for all $f \in L$, and*
3. *any $f \in L$ vanishes at infinity.*

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Let L be an Archimedean weakly truncated vector lattice L. Hence, there exists a positive weak unit w of L^u such that

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LEMMA

If $1 \in L$ then a nonzero linear functional ϕ on L is a truncation form if and only if

$$\phi(1) = 1 \quad \textit{and} \quad \phi(|f|) = |\phi(f)| \ \textit{ for all } f \in L.$$

Theorem (Shirota, 1952)

If X is a compact Hausdorff space, a linear functional ϕ on $C(X)$ is a truncation form if and only if $\phi = \delta_x$ for some $x \in X$, i.e., $\phi(f) = f(x)$ for all $f \in C(X)$.

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Theorem (Garrido-Jaramillo, 2004)

Let X be a Tychonoff space and ϕ be a linear functional on a vector sublattice of $C(X)$ such that $1 \in L$. Then ϕ is a truncated form on L if and only if there exists $u \in \beta X$ such that

$$\phi(f) = f^{\beta}(u) \quad \text{for all } f \in L,$$

where f^{β} is the unique extension of f to a continuous function from βX to $\omega\mathbb{R}$.

THEOREM

Let L be a truncated vector sublattice of $\mathbb{R}^X$ and ϕ be a linear functional on L. Then ϕ is a truncation form on L if and only if there exists a net (x_λ) of elements of X such that

$$\phi(f) = \lim f(x_\lambda) \text{ in } \mathbb{R} \quad \text{for all } f \in L.$$

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Extreme positive operators and truncations

Definition

Let A, B be two semiprime f-algebras. An operator $T : A \longrightarrow B$ is said to be **contractive** if

$$0 \leq Tf \leq I_B \quad \text{for all } f \in A \text{ with } 0 \leq f \leq I_A.$$

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Problem

We want to characterize the extreme points of $\mathcal{K}(A, B)$.

FACT

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Theorem

A linear operator $T : A \to B$ is an extreme point in $\mathcal{K}(A, B)$ if and only if T is a truncation homomorphism.

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1. *T is an extreme positive contraction.*
2. *$T(1 \wedge f) = 1 \wedge Tf$ for all $f \in C_0(X)$.*
3. *There exists a continuous function $\omega Y \xrightarrow{\tau} \omega X$ such that*

$$\tau(\omega) = \omega \quad \text{and} \quad Tf = f \circ \tau \text{ for all } f \in C_0(X).$$

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If the order ideal of a truncated Banach lattice L generated by the set of all positive fixed points of the truncation is norm-dense, we call L a **topologically truncated Banach lattice.**

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A complete normed truncated vector lattice is called a **truncated Banach lattice**.

Definition

If the order ideal of a truncated Banach lattice L generated by the set of all positive fixed points of the truncation is norm-dense, we call L a **topologically truncated Banach lattice.**

Theorem

A Banach lattice L is topologically truncated if and only if there exists a locally compact Hausdorff space X_L such that $C_0(X_L)$ is truncation (and so lattice) isomorphic with a norm-dense order ideal of L.

Definition

The **unit cone** of a truncated vector lattice L is the set

$$U(L) = \left\{ f \in L : |f|^* = |f| \right\}.$$

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The set $\mathcal{M}(L, M)$ of all almost Markov operators from L into M is convex.

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Fact

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Problem

What do the extreme points of $\mathcal{K}(L, M)$ look like?

THEOREM

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1. *T is an extreme almost Markov operator.*
2. *T is a truncation homomorphism.*
3. *T is continuous and there exists a continuous function $\omega X_F \xrightarrow{\tau} \omega X_E$ such that*

$$\tau(\infty) = \infty \quad \text{and} \quad Tf = f \circ \tau \text{ for all } f \in C_0(X_E).$$

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