Various characterizations of unbounded order convergence

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Order convergence in a vector lattice

For a net $(f_\alpha) \subset F$ and $f \in F$ TFAE (and is denoted by $f_\alpha \xrightarrow{o} f$):

- There is a net $(g_\beta) \subset F$ which decreases to 0_F and such that for every β there is α_β and such that $|f_\alpha - f| \leq g_\beta$, for $\alpha \geq \alpha_\beta$;
- There are $C, D \subset F$ with $\bigvee C = f = \bigwedge D$ and such that for every $c \in C$ and $d \in D$ there is $\alpha_{c,d}$ such that $f_\alpha \in [c, d]$, for $\alpha \geq \alpha_{c,d}$;
- f is the only element greater than every eventual lower bound, and smaller than every eventual upper bound;
- There is a family of order intervals in F whose intersection is $\{f\}$, and each of which contains a tail of the net;
- Same, but the family is directed downward.

Theorem 1 (Dabboorasad + Emelyanov + Marabeh, 2020)

Order convergence if topological iff $\dim F < \infty$.

If for some α_0 all extremums exist in $f = \bigvee\limits_{\alpha \geq \alpha_0} \bigwedge\limits_{\beta \geq \alpha} f_\beta = \bigwedge\limits_{\alpha \geq \alpha_0} \bigvee\limits_{\beta \geq \alpha} f_\beta$, then $f_\alpha \xrightarrow{o} f$. "$\Leftarrow$" holds if every order bounded set in F has a supremum.

Uo convergence in a vector lattice

Let F be Archimedean. A net $(f_\alpha) \subset F$ *unbounded order converges* to $f \in F$ (denoted $f_\alpha \xrightarrow{uo} f$) if $e \vee f_\alpha \wedge h \xrightarrow{o} e \vee f \wedge h$, for every $e \leq h$.

Theorem 2 (Papangelou, 1964-1965)

TFAE:

- $f_\alpha \xrightarrow{uo} f$;
- $|f_\alpha - f| \wedge h \xrightarrow{o} 0_F$, *for all* $h \geq 0_F$;
- f *is the only element with the property that for every* α*:*

$$\bigvee_{\beta \geq \alpha} (f \wedge f_\beta) = f = \bigwedge_{\beta \geq \alpha} (f \vee f_\beta)\,.$$

A sublattice $E \subset F$ is *order dense* if for every $f > 0_F$ there is $e \in E$ with $0_F < e \leq f$.

Theorem 3 (Implicit in Ellis, 1968)

UO convergence is topological iff F is atomic, i.e. embeds into $\mathbb{R}^X$, for some set X, as an order dense linear sublattice.

Homomorphisms and sublattices

Theorem 4 (Turan + Altın + Gürkök, 2022, B., 2023)

A lattice homomorphism is order continuous iff it is UO-continuous.

A sublattice $E \subset F$ is *regular* if $A \subset E$: $\bigwedge_E A = 0_F \Rightarrow \bigwedge_F A = 0_F$.

Every order dense sublattice as well as every ideal are regular.

Corollary 1 (Papangelou, 1964, rediscovered by Abramovich + Sirotkin & Gao + Troitsky + Xanthos)

For a sublattice $E \subset F$ TFAE:

- *E is regular;*
- *$[0_E, e]$ (interval in E) order-homeomorphically embeds into F, for every $e \in E$;*
- *the inclusion map is UO-continuous;*
- *the inclusion map is a UO-homeomorphic embedding.*

Criteria for convergence

Theorem 5 (Troitsky + B., 2022)

In $\mathcal{C}(X)$ we have $f_\alpha \xrightarrow{uo} \mathbb{0}$ iff for every open $U \neq \varnothing$ and $\varepsilon > 0$ there is an open $\varnothing \neq V \subset U$ and α_0 such that $|f_\alpha|_{|V} \leq \varepsilon$, for $\alpha \geq \alpha_0$.

Corollary 2 (Kantorovich + Pinsker + Vulikh, 1950; van der Waalt, 2018; Troitsky + B., 2022)

If $f_\alpha \xrightarrow{uo} f$, then $f_\alpha(x) \to f(x)$, for all x outside of a meager set. The converse holds if X is Baire and the net is countable.

Note that in $L_p(\mu)$ uo convergence of sequences = a.e. convergence.

Theorem 6 (B., 2023)

$0_F \leq f_\alpha \xrightarrow{uo} 0_F$ iff for every $h > 0_F$ there is $e \in (0_F, h]$ and α_0 such that $(f_\alpha - h)^+ \perp e$, for all $\alpha \geq \alpha_0$.

Uo convergence via BANDS

For $f \in F$ let $\mathcal{B}(f) := \{f\}^{dd}$. Note that the set $\mathcal{B}_F$ of all bands in F is a Boolean algebra.

Corollary 3 (B., 2023)

If g is a weak unit of F, then $0_F \leq f_\alpha \xrightarrow{uo} 0_F$ *iff*

$$\mathcal{B}(f_\alpha - \varepsilon g)^+ \xrightarrow{o} \{0_F\}$$

(order convergence in $\mathcal{B}_F$), for every $\varepsilon > 0$.

Proposition 1 (Troitsky + B., 2022)

In a Boolean algebra B we have $B \ni b_\alpha \xrightarrow{o} 0_B$ iff for every $c > 0_B$ there is $d \in (0_B, c]$ and α_0 such that $b_\alpha \perp d$, for all $\alpha \geq \alpha_0$.

Let μ be a finite measure. Then, in $F = L_p(\mu)$ the algebra $\mathcal{B}_F$ consists of classes of measurable sets up to μ-negligible sets.

Corollary 4 (B., 2023)

If μ is a finite measure, then $L_p(\mu) \ni f_\alpha \xrightarrow{uo} \mathbb{O}$ iff for every U with $\mu(U) > 0$ and $\varepsilon > 0$ there is $V \subset U$ with $\mu(V) > 0$ and α_0 such that $|f_\alpha|_{|V} \leq \varepsilon$, for $\alpha \geq \alpha_0$.

How is this connected with Egoroff's theorem?

Let η be a "nice enough" locally solid convergence on $\mathcal{B}_F$. Then, it generates two locally solid convergences on F by:

- $f_\alpha \xrightarrow{h\eta} 0_F$ if $\mathcal{B}(f_\alpha) \xrightarrow{\eta} \{0_F\}$ (this is non-linear convergence).
- $f_\alpha \xrightarrow{\ell\eta} 0_F$ if $\mathcal{B}(f_\alpha - \varepsilon g)^+ \xrightarrow{\eta} \{0_F\}$, for every $\varepsilon > 0$.

For example, in $F = L_p(\mu)$, then μ induces a convergence η on $\mathcal{B}_F$ given by a metric on $\mathcal{B}_F$ by $\mu(A \triangle B)$, where A, B are representatives.

Then $h\eta$ is given by $\rho(f) := \mu(\mathrm{supp} f)$, while $\ell\eta$ is the convergence in measure.

Connection with minimal topologies

Theorem 7 (Sarymsakov + Rubinstein + Chillin & Weber, 70s)

There is at most one Hausdorff order continuous locally solid topology on a Boolean algebra.

Question 1

If such topology exist, is it equal to the order topology?

For example, in the measure algebra this topology is precisely the one given by the metric generated by μ.

Theorem 8 (B., 2023)

F admits an order continuous Hausdorff topology iff $\mathcal{B}_F$ does. Moreover, a locally solid Hausdorff τ on F is minimal iff $\tau = \ell\pi$, where π is the order continuous topology on $\mathcal{B}_F$.

THANK YOU!