Tensor product of Riesz subspaces

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based on joint work with Damla Yaman and Omer Gok
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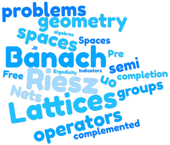
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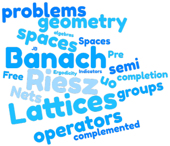
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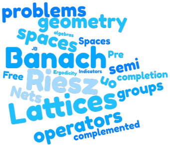
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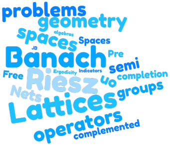
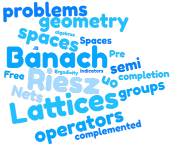


The Positivity community



The Positivity community

COSA Conference, Save the dates!
February 5 to 9, 2024

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Historical facts

- Martinez, J. (1972). Tensor products of partially ordered groups. Pacific Journal of Mathematics, 41(3), 771-789.

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- Grobler, J. J., & Labuschagne, C. C. A. (1989). An f-algebra approach to the Riesz tensor product of Archimedean Riesz spaces. Quaestiones Mathematicae, 12(4), 425-438.

Tensor products

Theorem (Fremlin,1972)

Let E and F be Archimedean Riesz spaces. Then there is an Archimedean Riesz space G and a Riesz bimorphism $\varphi\colon E \times F \to G$ such that

(i) whenever H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G \to H$ such that $T\varphi = \psi$;

(ii) φ induces an embedding $\hat{\varphi}\colon E \otimes F \to G$;

(iii) (ru-D) $\hat{\varphi}[E \otimes F]$ is dense in G in the sense that for every $w \in G$, there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\epsilon > 0$, there is an element $v \in \hat{\varphi}[E \otimes F]$ such that $|w - v| \leq \epsilon\hat{\varphi}(x_0 \otimes y_0)$;

(iv) if $w > 0$ in G, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < \hat{\varphi}(x \otimes y) \leq w$.

This essentially unique Archimedean Riesz space G is called the Fremlin tensor product of E and F and is denoted by $E\bar{\otimes}F$.

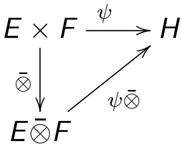
Tensor Products

Fremlin tensor product

u-D)$_+$ For $w \in G_+$, there exists an element $(x, y) \in E_+ \times F_+$ such that for every $\epsilon > 0$, there exists an element $v \in (E_x)_+ \otimes (F_y)_+$ with $|w - v| \leq \epsilon \hat{\varphi}(x \otimes y)$

PUM) Let G be a relatively uniformly complete Archimedean Riesz space and $\psi : E \times F \to G$ be a positive linear mapping. Then there exists a unique positive linear mapping $\tau : E \overline{\otimes} F \to G$ such that $\tau \circ \varphi = \psi$.

(B) If $h \in E \overline{\otimes} F$, there exists an element $(x, \ y) \in E_+ \times F_+$ such that $|h| \leq x \otimes y$.



Continuity

Theorem (MABA, OG, DY,2022)

Let E and F be two Archimedean Riesz spaces and let σ be the map defined by:

$$\begin{array}{rccc} \sigma : & E \times F & \longrightarrow & E\bar{\otimes}F \\ & (x, y) & \longmapsto & x \otimes y. \end{array}$$

Then σ is order continuous.

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- Buskes, G.J.H.M., Wickstead, A.W. Tensor Products of f-algebras. Mediterr. J. Math. 14, 63 (2017).
- Azouzi, Y., Ben Amor, M. A., & Jaber, J. (2018). The tensor product of f-algebras. Quaestiones Mathematicae, 41(3), 359-369. 10 (2017).
- Ben Amor, M. A. (2020). The riesz tensor product ofd-algebras. Quaestiones Mathematicae.

Tensor product of Riesz subspaces

Problem (Azouzi,Ben Amor, Jaber,2018)

- Is the tensor product of ideals an ideal?
- Is the tensor product of bands a band?

Tensor product of ideals and bands

Problem (Azouzi,Ben Amor, Jaber,2018)

- Is the tensor product of ideals an ideal?(negative, Buskes and Thorn's counter-example)
- Is the tensor product of bands a band?(Still open!)

Tensor product of ideals and bands

Problem (Azouzi,Ben Amor, Jaber,2018)

- Is the tensor product of ideals an ideal?(negative, Buskes and Thorn's counter-example)
- Is the tensor product of bands a band?(Still open!)

The Buskes-Thorn Counterexample, 2022

A function $p : [0, \infty) \to \mathbb{R}$ is said to be a piecewise polynomial if there are $n \in \mathbb{N}$ and $t_1, ..., t_n \in [0, \infty)$ such that $t_1 < t_2 < ... < t_n$ and p is a polynomial function on $[t_n, \infty)$ and $[t_i, t_{i+1}]$ for each $i = 1, ..., n-1$. Suppose $E = F = PP([0, \infty))$, the Archimedean Riesz space of piecewise polynomials on $[0, \infty)$. Pick the piecewise polynomials $p(x) = x$ and $q(y) = y$ in $PP([0, \infty))$. Then $E_p \bar{\otimes} E_q$ is not an ideal in $E \bar{\otimes} F$.

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Dedekind complete tensor product

Theorem (Grobler, 2022)

Let E, F be two Archimedean RS. The t.p. $E\overline{\otimes}_\delta F$ is a Dedekind complete VL with the following properties:

(D3) $E \otimes F$ is a vector subspace of $E\overline{\otimes} F$ and $E\overline{\otimes} F$ is a Riesz subspace of $E\overline{\otimes}_\delta F$.

(D5) If G is a Dedekind complete VL and $\psi : E \times F \to G$ is an o-continuous Riesz bimorphism, then there exists an o-continuous Riesz homomorphism $\tau : E\overline{\otimes}_\delta F \to G$ such that $\tau \circ \sigma = \psi$.

(D4) The Dedekind complete Riesz subspace generated in $E\overline{\otimes}_\delta F$ by $E \otimes F$ is equal to $E\overline{\otimes}_\delta F$.

Dedekind complete tensor product

(OD) For every $0 < h \in E\overline{\otimes}_\delta F$ there exists an element $(x,\ y) \in E_+ \times F_+$ such that $0 < x \otimes y \leq h$.

(B) If $h \in E\overline{\otimes}_\delta F$, there exists an element $(x,\ y) \in E_+ \times F_+$ such that $|h| \leq x \otimes y$.

PUM) Let G be a Dedekind complete Archimedean RS and let $\psi : E \times F \to G$ be an o-continuous positive bilinear mapping. Then there exists a unique o-continuous positive linear mapping $\tau : E\overline{\otimes}_\delta F \to G$ such that $\tau \circ \sigma = \psi$.

Tensor product of ideals

Lemma (MABA, OG, DY,2022)

Let E and F be Riesz spaces, $x,\ x'$ be strictly positive elements in E and $y,\ y'$ be strictly positive elements in F. If $x \otimes y \leq x' \otimes y'$, then x belongs in the principal ideal generated by x', and y belongs in the principal ideal generated by y'.

Tensor product of ideals

Lemma (MABA, OG, DY,2022)

Let E and F be Riesz spaces, $x,\ x'$ be strictly positive elements in E and $y,\ y'$ be strictly positive elements in F. If $x \otimes y \leq x' \otimes y'$, then x belongs in the principal ideal generated by x', and y belongs in the principal ideal generated by y'.

Proposition (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f be a positive element in F. Let E_e and E_f be the principal ideals generated by e and f, respectively. Then $E_e \bar{\otimes} F_f$ is an order dense Riesz subspace of $(E \bar{\otimes} F)_{e \otimes f}$, where $(E \bar{\otimes} F)_{e \otimes f}$ is the principal ideal in $(E \bar{\otimes} F)$ generated by $e \otimes f$.

Dedekind complete tensor product of principal ideals and ideals

Lemma (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f be a positive element in F, E_e be the principal ideal generated by e and E_f be the principal ideal generated by f. Then

$$(E\bar{\otimes}_\delta F)_{e\otimes f} = \overline{(E\bar{\otimes}F)_{e\otimes f}}^{\delta}.$$

Dedekind complete tensor product of principal ideals and ideals

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Dedekind complete tensor product of principal ideals and ideals

Corollary (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f in F. Let E_e be the principal ideal generated by e and F_f by f. If $E\bar{\otimes}F$ is Dedekind complete then $E_e\bar{\otimes}_\delta F_f$ is an ideal in $E\bar{\otimes}F$.

Dedekind complete tensor product of principal ideals and ideals

Theorem (MABA, OG, DY,2022)

Let E and F be RS and A and B be two ideals of E and F respectively. Then $A\bar{\otimes}_\delta B$ is an ideal in $E\bar{\otimes}_\delta F$.

Corollary (MABA, OG, DY,2022)

Let E and F be RS and A and B be ideals of E and F respectively. If $E\bar{\otimes}F$ is Dedekind complete then $A\bar{\otimes}_\delta B$ is an ideal in $E\bar{\otimes}F$.

Contents

Preliminary result

Lemma(MABA, OG, DY, 2022)

Let u be a positive element in a Riesz space E. Then the principal band B_u generated by u in E is order dense and majorizing in B_u^δ, the principal band generated by u in the Dedekind completion E^δ of E.

Tensor product of bands

Theorem (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f be a positive element in F. Let B_e be the principal band generated by e and B_f be the principal band generated by f, then $B_e \bar{\otimes} B_f$ is an order dense Riesz subspace in the principal band in $E \bar{\otimes} F$ generated by $e \otimes f$.

Tensor product of bands

Theorem (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f be a positive element in F. Let B_e be the principal band generated by e and B_f be the principal band generated by f, then $B_e \bar{\otimes} B_f$ is an order dense Riesz subspace in the principal band in $E \bar{\otimes} F$ generated by $e \otimes f$.

Proposition (MABA, OG, DY,2022)

Let e and f be weak order units of the Riesz spaces E and F, respectively. Then $e \otimes f$ is a weak unit in $E \bar{\otimes} F$.

Dedekind complete tensor product of principal bands

Lemma (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f in F. Let B_e and B_f be the principal bands generated by e and f respectively. Then for every positive element v in $(e \otimes f)^{dd}$, there exists a positive element u in $B_e \bar{\otimes}_\delta B_f$ such that

$$0 \leq v \leq u.$$

That is $B_e \bar{\otimes}_\delta B_f$ is majorizing in $(e \otimes f)^{dd}$.

Dedekind complete tensor product of principal bands

Lemma (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f in F. Let B_e and B_f be the principal bands generated by e and f respectively. Then for every positive element v in $(e \otimes f)^{dd}$, there exists a positive element u in $B_e \bar{\otimes}_\delta B_f$ such that

$$0 \leq v \leq u.$$

That is $B_e \bar{\otimes}_\delta B_f$ is majorizing in $(e \otimes f)^{dd}$.

Theorem (MABA, OG, DY,2022)

Let E and F be RS. Let e be a positive element in E, and f in F. Let B_e be the principal band generated by e and B_f by f. Then

$$B_e \bar{\otimes}_\delta B_f = (e \otimes f)^{dd}.$$

Dedekind complete tensor product of projection bands

Corollary (MABA, OG, DY,2022)

Let E and F be RS with weak units e and f respectively. Then if A_1 and A_2 are projection bands in E and F respectively then $A_1 \bar{\otimes}_\delta A_2$ is a projection band in $E \bar{\otimes}_\delta F$.

Dedekind complete tensor product of projection bands

Corollary (MABA, OG, DY,2022)

Let E and F be RS with weak units e and f respectively. Then if A_1 and A_2 are projection bands in E and F respectively then $A_1 \bar{\otimes}_\delta A_2$ is a projection band in $E \bar{\otimes}_\delta F$.

Corollary (MABA, OG, DY,2022)

Let E and F be two Dedekind complete RS with weak units e and f respectively. If A_1 and A_2 are bands in E and F, respectively, then $A_1 \bar{\otimes}_\delta A_2$ is a projection band in $E \bar{\otimes}_\delta F$.

Contents

Conditional expectation

If E is a Riesz space with weak order unit e, then $T : E \to E$ is a conditional expectation operator on E whenever T is :

1. a projection
2. positive
3. order continuous
4. $Te = e$
5. its range $\mathcal{R}(T) = \{Tf \mid f \in E\}$, is a Dedekind complete Riesz subspace of E

Analogy

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space	The conditional Riesz triple(E, e, T)
The function $\mathbb{1}$	The weak order unit e
$A \subset \Omega$: event	P: order projection
χ_A: indicator function	Band projection
$TP_{(f-\epsilon e)^+}$	$\mathbb{P}(X \geq \epsilon)$

Fubini Theorem

Theorem (MABA, Watson, Kuo, Yaman, 2023)

Let E and F be Dedekind complete Riesz spaces and let T and S be conditional expectation operators defined on E and F, respectively. Let e be a weak order unit in E and let f be a weak order unit in F.

$$T\bar{\otimes}S = (T\bar{\otimes}\mathfrak{I}_F) \circ (\mathfrak{I}_E\bar{\otimes}S)$$

where $\mathfrak{I}_E$ and $\mathfrak{I}_F$ are identity operators on E and F, respectively.

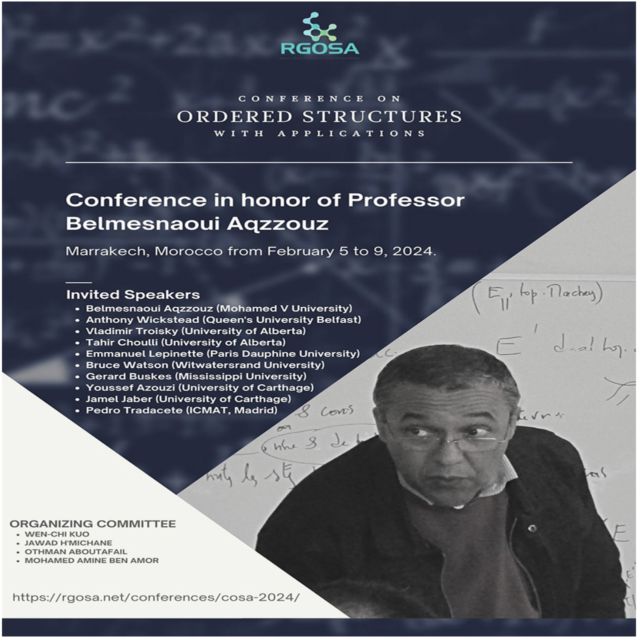
Weak Mixing Theorem

Theorem (MABA, Watson, Kuo, Homann, 2022)

In a conditional expectation preserving system (E, T, S, e), with T strictly positive, the following are equivalent.

1. (E, T, S, e) is conditionally weak mixing.
2. $\left(\overline{E\overline{\otimes}E_1}^{\delta}, T \otimes_\delta T_1, S \otimes_\delta S_1, e \otimes e_1\right)$ is ergodic for each ergodic (E_1, T_1, S_1, e_1) with T_1 strictly positive.
3. $\left(\overline{E\overline{\otimes}E}^{\delta}, T \otimes_\delta T, S \otimes_\delta S, e \otimes e\right)$ is ergodic.

Thank you for your attention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• Anthony Wickstead (Queen's University Belfast)
• Vladimir Troisky (University of Alberta)
• Tahir Choulli (University of Alberta)
• Emmanuel Lepinette (Paris Dauphine University)
• Bruce Watson (Witwatersrand University)
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