Positivity of infinite-dimensional linear control systems

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Positivity XI

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Motivation

Let us consider the following system of transport equations on a network:

$$\begin{cases} \dfrac{\partial}{\partial t} z_j(t,x,v) = v\dfrac{\partial}{\partial x} z_j(t,x,v) + q_j(x,v).z_j(t,x,v), & t \geq 0, \quad (x,v) \in \Omega_j, \\ z_j(0,x,v) = f_j(x,v) \geq 0, & (x,v) \in \Omega_j, \\ \imath_{ij}^{out} z_j(t,1,\cdot) = \mathrm{w}_{ij} \sum_{k\in\mathcal{M}} \imath_{ik}^{in} \mathbb{J}_k(z_k)(t,0,\cdot) + \sum_{l\in\mathcal{N}_c} \mathrm{b}_{il} u_l(t,.), & t \geq 0, \end{cases}$$

for $i \in \{1,\dots,N\} := \mathcal{N}$, $j \in \{1,\dots,M\} := \mathcal{M}$ and $l \in \{1,\dots,n\} := \mathcal{N}_c$ with $M \geq N \geq n$, where we set $\Omega_j := [0,l_j] \times [v_{\min}, v_{\max}]$, $l_j > 0$. Here, the scattering operators $\mathbb{J}_j$ are given by

$$(\mathbb{J}f)_j(x,v) = \int_{v_{\min}}^{v_{\max}} \ell_j(x,v,v') f_j(x,v') dv', \qquad (x,v) \in \Omega_j,\ f_j \in L^p(\Omega_j),$$

where $\ell_j \in L^\infty(\Omega_j \times [v_{\min}, v_{\max}])$ for every $j \in \mathcal{M}$.

We consider the following size-structured population model with delayed birth process:

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,s) + \dfrac{\partial}{\partial s} q(s) z(t,s) = -\mu(s) z(t,s). & t \geq 0, \quad s \in (0, s^*), \\ z(0,s) = f(s) \geq 0, \quad z(\theta, s) = \varphi(\theta, s) \geq 0, \quad \theta \in [-r, 0], & s \in (0, s^*), \\ g(0) z(t,0) = \int_0^{s^*} \int_{-r}^0 \beta(s) z(t+\theta, s) d\eta(\theta) ds + bu(t), & t \geq 0. \end{cases}$$

Here, $z(t,s)$ represents the population density of certain species of size $s \in (0, s^*)$ at time $t \geq 0$, where $s^* > 0$ is the maximum size of of any individual in the population. The function $q(s)$ is the growth rate of size over time and the size dependent functions β and μ denote the fertility and mortality, respectively. We denotes by b the boundary control operator, u define the control functions and (f, φ) are the initial distributions of our target population.

Motivation

Can be transformed as the following perturbed boundary value control system

$$\begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x \geq 0, \\ Gz(t) = \Gamma z(t) + Ku(t), & t > 0, \end{cases}$$

where the state variable $z(.)$ takes values in a Banach lattice X and the control function $u(\cdot)$ is given in the Banach space $L^p([0, +\infty); U)$.

- The maximal (differential) operator $A_m : D(A_m) \subset X \to X$ is closed and densely defined.
- K is a bounded linear operator from U into ∂X (both are Banach lattices).
- $G, \Gamma : D(A_m) \to \partial X$ are linear continuous trace operators.

Boundary control positive systems

Consider the following boundary control system

$$(\mathrm{CS})\begin{cases}\dot{z}(t)=A_m z(t), & z(0)=x, \quad t>0,\\ Gz(t)=v(t), & \quad t>0,\end{cases}$$

for $x \in X_+$ and $v \in L^p_+(\mathbb{R}_+;\partial X)$, where we recall that $X,\partial X$ are Banach lattices.

Main Assumptions:

H1. $A := (A_m)_{|D(A)}$ with $D(A) = \ker G$, generates a strongly continuous positive semigroup $\mathbb{T} := (T(t))_{t\geq 0}$ on X;

H2. $\mathrm{Range}\,(G) = \partial X$ and Γ is positive.

Boundary control positive systems

For any $\mu \in \rho(A)$, we have

$$D(A_m) = D(A) \oplus \ker(\mu I_X - A_m)$$

and the following inverse (called the Dirichlet operator associated with (G, A))

$$D_\mu := \left(G_{|\ker(\mu I_X - A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X)$$

exists. So,it makes sense to define the following operator

$$B := (\mu I_X - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1,A}), \qquad \mu \in \rho(A).$$

The extrapolation space associated with X and A, denoted by $X_{-1,A}$, is the completion of X with respect to the norm $\|x\|_{-1} := \|R(\mu, A)x\|$ for $x \in X$ and some $\mu \in \rho(A)$.

$$X_+ = (X_{-1,A})_+ \cap X.$$

A. Bátkai, B. Jacob, J. Voigt, and J. Wintermayr, Perturbation of positive semigroups on *AM*-spaces. Semigroup Forum. **96** (2018) 33-347.

G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213-229.

Boundary control positive systems

The system (CS) is reformulated as follows

$$\dot{z}(t) = A_{-1}z(t) + Bv(t), \quad z(0) = x, \quad t \geq 0.$$

- An integral solution of the above equation is given by: $z(t) = T(t)x + \Phi_t v \ \in X_{-1}$, for any $t \geq 0,\ x \in X$ and $v \in L^p([0, +\infty), \partial X)$, where

$$\Phi_t v := \int_0^t T_{-1}(t-s)Bv(s)ds \in X_{-1,A} \qquad (*).$$

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In particular, $z(\cdot) \in C([0,+\infty), X)$ for all $x \in X$ and $v \in L^p([0,+\infty), \partial X)$.

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$$\Phi_t v := \int_0^t T_{-1}(t-s)Bv(s)ds \qquad (*).$$

- An operator $B \in \mathcal{L}(\partial X, X_{-1})$ is called L^p-admissible control operator for A if, for some $\tau > 0$, the operator $\mathrm{Range}\,\Phi_\tau \subset X$.

G. Weiss, Admissibility of unbounded control operators. SIAM J. Control Optim. **27** (1989), 527-545.

We have the following characterization of the well-posedness of (CS).

Proposition (Yassine, 2022)

Let $X, \partial X$ be Banach lattices, let $(A, D(A))$ generate a strongly continuous semigroup $\mathbb{T}$ on X and $B \in \mathcal{L}(\partial X, X_{-1})$. Then for every $x \in X$ and $v \in L^p_{loc}(\mathbb{R}_+; \partial X)$, the system (CS) has a unique solution $z(\cdot) \in C(\mathbb{R}_+; X)$ if one of the following equivalent assertions holds.

- (*i*) for any $x \in X_+$ and $v \in L^p_+(\mathbb{R}_+; \partial X)$, the solution $z(\cdot)$ of (CS) remains in X_+.
- (*ii*) $\mathbb{T}, B$ are positive and for some $\tau > 0$, $\Phi_\tau v \in X_+$ for all $v \in L^p_+(\mathbb{R}_+; \partial X)$.

Lemma

Let $X, \partial X$ be Banach lattices and let $\mathbb{T}$ be a positive C_0-semigroup on X. Then,

$$B \geq 0 \quad \text{iff} \quad D_\mu := R(\mu, A_{-1})B \geq 0, \quad \forall \mu > s(A) \quad \text{iff} \quad \Phi_t \geq 0, \quad \forall t > 0.$$

Boundary control positive systems

Remarks:

- Let $\mathfrak{B}_p(\partial X, X, \mathbb{T})$ denote the vector space of all L^p-admissible control operators B, which is a Banach space with the norm

$$\|B\|_{\mathfrak{B}_p} := \sup_{\|v\|_{L^p([0,\tau];\partial X)} \leq 1} \left\| \int_0^\tau T_{-1}(\tau - s)Bv(s)ds \right\|,$$

  where $\tau > 0$ is fixed.
- One could relax the definition of L^p-admissible positive control operators to

$$\|\Phi_t v\|_X \leq \eta(\tau)\|v\|_{L^p([0,\tau];\partial X)},$$

  for all $v \in L^p_+([0,\tau], \partial X)$ and for some $\tau \geq 0$.
- $\mathfrak{B}_{p,+}(\partial X, X, \mathbb{T})$ is the positive convex cone in $\mathfrak{B}_p(\partial X, X, \mathbb{T})$.
- It generates the order: for $B, B' \in \mathfrak{B}_p(\partial X, X, \mathbb{T})$, we have $B' \leq B$ whenever $B - B' \in \mathfrak{B}_{p,+}(\partial X, X, \mathbb{T})$.

(Yassine, 2022)

Let $X, \partial X$ be Banach lattices, let A be a densely defined resolvent positive operator such that

$$\|R(\mu_0, A)x\| \geq c\|x\|, \qquad x \in X_+, \qquad (**)$$

for some $\mu_0 > s(A)$ and constant $c > 0$. Then, $B \in \mathcal{L}(\partial X, X_{-1})$ is a positive L^p-admissible control operator for A. Furthermore, for $p > 1$, B is of zero-class.

Sketch of the proof:

- For any positive constant function $v \in \partial X$, $\Phi_\tau^A v \in X_+ \, (= X \cap (X_{-1,A})_+)$.
- $c\|\Phi_\tau^A v\| \leq \|R(\mu_0, A)\Phi_\tau^A v\| \leq \kappa(\tau)\|v\|_{L^p([0,\tau];\partial X)}$ with $\kappa(\tau) := Me^{\omega\tau}\tau^{\frac{p-1}{p}}\|R(\mu_0, A_{-1})B\|$

C. J. K. Batty and D. W. Robinson, Positive one-parameter semigroups on ordered Banach space. Acta Appl. Math. **2** (1984) 221-296.

W. Arendt, Resolvent positive operators. Proc. London Math. Soc. **54** (1987) 321-349.

Boundary control positive systems

Desch-Schappacher perturbation, (Yassine, 2022)

Let $(A, D(A))$ be a densely defined resolvent positive operator satisfying the inverse estimate $(**)$ and let $B \in \mathcal{L}(X, X_{-1})$ be a positive operator. Then $(A_{-1} + B)_{|X}$ generates a positive C_0-semigroup on X and

$$s((A_{-1} + B)_{|X}) = w_0((A_{-1} + B)_{|X})$$

Sketch of the proof:

- $(A_{-1} + B)_{|X}$ is resolvent positive.
- $(A_{-1} + B)_{|X}$ satisfies the inverse estimate $(**)$.

Application

Let us consider a usual definition when dealing with delay equations. Namely, for a function $z \in L^1([-r,\infty))$ and for $t \geq 0$, we define $z_t \in L^1([-r,0])$ as $z_t(\theta) = z(t+\theta)$. In particular, notice that $z_0 = z_{|_{[-r,0]}}$. Also, we underline that the function $t \mapsto z_t$ is the solution of the boundary equation

$$\begin{cases} \frac{\partial}{\partial t} v(t,\theta) = \frac{\partial}{\partial \theta} v(t,\theta), & t \geq 0,\ \theta \in [-r,0], \\ v(t,0) = z(t), \quad v(0,\theta) = \varphi(\theta), & t \geq 0,\ \theta \in [-r,0], \end{cases}$$

Let us consider the operators $Q_m : D(Q_m) \subset L^1([-r,0]) \to L^1([-r,0]) := Y$ and $Q : D(Q) \subset Y \to Y$ defined by

$$Q_m\varphi = \partial_\theta \varphi, \qquad D(Q_m) = W^{1,1}([-r,0]),$$
$$Q\varphi = (Q_m)_{|D(Q)}, \quad \text{with} \quad D(Q) = \ker\{I_Y \otimes \delta_0\}.$$

Note that the operator Q generates the left shift semigroup $(S(t))_{t\geq 0}$ on Y given by

$$(S(t)\varphi)(\theta) = \begin{cases} 0, & -t \leq \theta \leq 0, \\ \varphi(t+\theta), & -r \leq \theta \leq -t, \end{cases}.$$

Application

Using the additivity of the norm on the positive cone Y_+ and Fubini's theorem we have

$$\begin{aligned}\|R(\lambda, Q)\varphi\|_Y &= \int_{-r}^{0} \|(R(\lambda, Q)\varphi)(\theta, \cdot)\| d\theta \\ &= \int_{-r}^{0} (R(\lambda, Q)\varphi)(\theta) d\theta \\ &= \int_{-r}^{0} \frac{1}{\lambda}(e^{\lambda\sigma} - e^{-\lambda r})e^{-\lambda\sigma}\varphi(\sigma, s) d\sigma \\ &\geq \int_{-r}^{0} \frac{1}{\lambda}(e^{\lambda\sigma} - e^{-\lambda r})\varphi(\sigma, s) d\sigma,\end{aligned}$$

for all $\lambda > 0$ and $\varphi \in Y_+$. It follows from the Mean Value Theorem that there exists $\alpha \in (-r, 0)$ such that

$$\int_{-r}^{0} \frac{1}{\lambda} e^{\lambda\sigma}\varphi(\sigma) d\sigma = \frac{1}{\lambda} e^{\lambda\alpha} \int_{-r}^{0} \varphi(\sigma) d\sigma, \qquad \forall\, 0 \leq f \in C([-r, 0]).$$

Thus, by density,

$$\|R(\lambda, Q)\varphi\|_Y \geq \frac{1}{\lambda}(e^{\lambda\alpha} - e^{-\lambda r})\|\varphi\|_Y, \qquad \forall\, \lambda > 0,\ \varphi \in Y_+.$$

Thank you