



Irreducibility of eventually positive semigroups

Positivity XI (CA18232 Session)

10th July 2023

Sahiba Arora (joint work with **Jochen Glück** (Wuppertal))

Throughout, we consider the abstract Cauchy problem

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Right-shift semigroup

On $L^2(0, 1)$, let $(e^{tA}f)(s) := f(s - t)$ for $s - t > 0$ and 0 otherwise.

Positive: $e^{tA} \geq 0$ for all $t \geq 0$ (Vast theory).

Motivation

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Eventually positive: $e^{tA} \geq 0$ for all $t \geq t_0$ (Developing theory).

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- By-product: theory only applicable if $e^{tA} \gg 0^1 \forall t \geq t_0$.

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Irreducible: Nontrivial² closed ideals aren't invariant (Widely studied).

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²Nontrivial : \Leftrightarrow non-zero and proper

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Natural idea: Study irreducibility for eventually positive semigroups.

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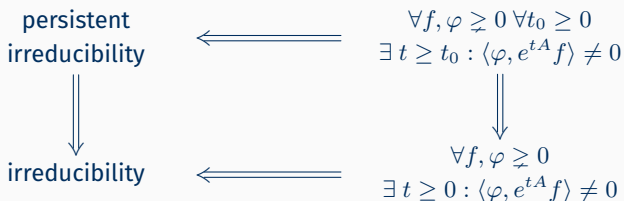
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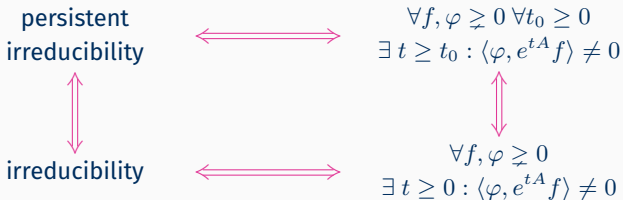


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For **positive** semigroups,

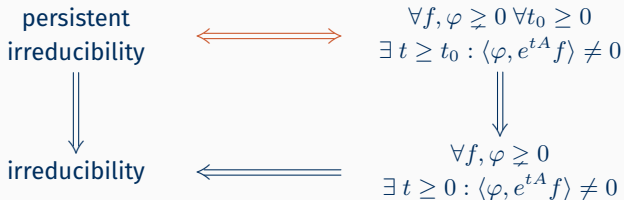


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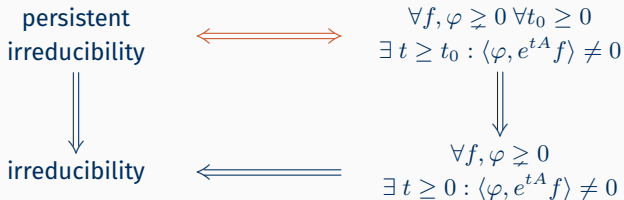


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Let $(e^{tB})_{t \geq 0}$: right-shift and U : rademacher ONB \mapsto standard ONB:

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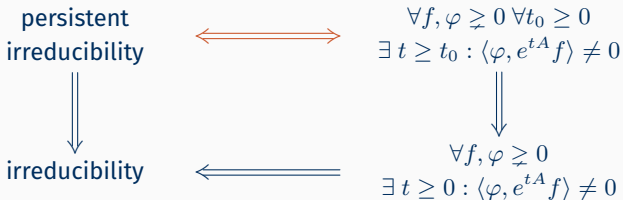
Then $(e^{tA})_{t \geq 0}$ is irreducible but not persistently irreducible.

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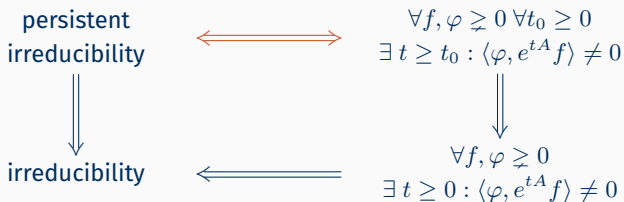
Connection with (eventual) strong positivity

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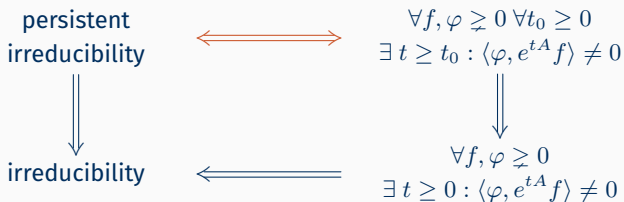
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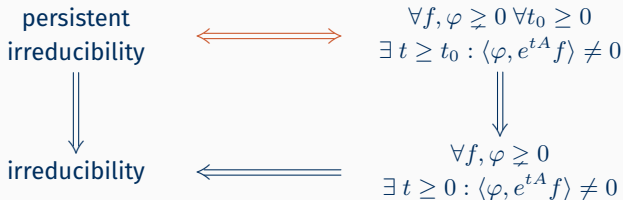
$e^{tA} \gg 0$ for all $t > t_0 \Rightarrow$ persistently irreducible.

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Converse

Positive + irreducible + analytic $\Rightarrow e^{tA} \gg 0$ for all $t > 0$.

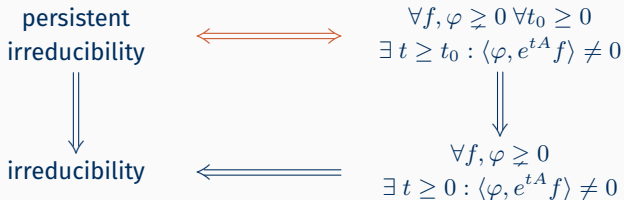
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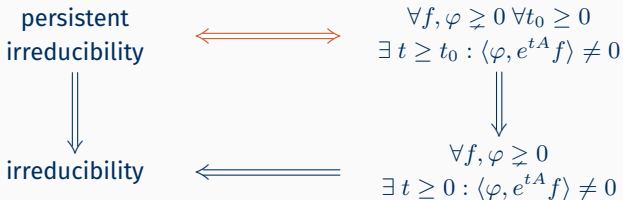
Positive + irreducible on $C_0(L) \Rightarrow \sigma(A) \neq \emptyset$.

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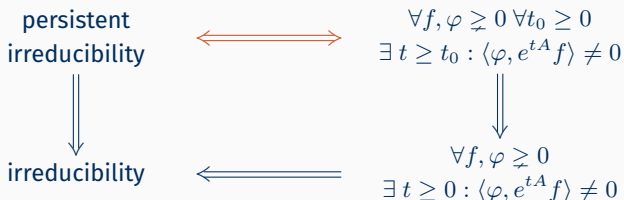
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Positive + irreducible + $e^{t_0 A}$ compact $\Rightarrow \sigma(A) \neq \emptyset$.

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Open question

Eventual positive + persistent irreducible + $e^{t_0 A}$ compact $\Rightarrow \sigma(A) \neq \emptyset$?