TECHNISCHE UNIVERSITÄT DRESDEN

Bereich Mathematik und Naturwissenschaften Fakultät Mathematik, Institut für Analysis

Irreducibility of eventually positive semigroups

Positivity XI (CA18232 Session) 10th July 2023

Sahiba Arora (Joint work with Jochen Glück (Wuppertal))

Setting

Throughout, we consider the abstract Cauchy problem

$$\dot{u}(t) = Au(t) \quad (t \geq 0), \qquad u(0) = f;$$

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Right-shift semigroup

On $L^2(0,1)$, let $(e^{tA}f)(s) := f(s-t)$ for $s - t > 0$ and 0 otherwise.

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- By-product: theory only applicable if $e^{tA} \gg 0$[1] $\forall\, t \geq t_0$.

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Irreducible: Nontrivial[2] closed ideals aren't invariant (Widely studied).

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[2] Nontrivial $:\Leftrightarrow$ non-zero and proper

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Natural idea: Study irreducibility for eventually positive semigroups.

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In general,

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For positive semigroups,

persistent irreducibility $\Longleftrightarrow$ $\forall f, \varphi \gneq 0 \; \forall t_0 \geq 0 \;\; \exists\, t \geq t_0 : \langle \varphi, e^{tA} f \rangle \neq 0$

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Let $(e^{tB})_{t \geq 0}$: right-shift and U : rademacher ONB $\mapsto$ standard ONB:

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\ell^2 & \xrightarrow{\;e^{tA}\;} & \ell^2 \\
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Then $(e^{tA})_{t \geq 0}$ is irreducible but not persistently irreducible.

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$e^{tA} \gg 0$ for all $t > 0 \Rightarrow$ irreducible.
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Converse

Positive + irreducible + analytic $\Rightarrow e^{tA} \gg 0$ for all $t > 0$.
Eventual positive + persistent irreducible + analytic $\Rightarrow e^{tA} \gg 0 \; \forall\, t > t_0$.

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Positive + irreducible + $e^{t_0 A}$ compact $\Rightarrow \sigma(A) \neq \emptyset$.

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Open question

Eventual positive + persistent irreducible + $e^{t_0 A}$ compact $\Rightarrow \sigma(A) \neq \emptyset$?