Notes on oscillating semigroups

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HOUSTON JOURNAL OF MATHEMATICS
Volume 21, No. 2, 1995

A CHARACTERIZATION OF WEAKLY OSCILLATING C_0–SEMIGROUPS

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Communicated by the editors.

Definition 1.1. A function $u \in C(\mathbb{R}_+, \mathbb{R})$ *oscillates* if u has arbitrarily large zeros, i. e., for all $t_1 > 0$ there exists $t_2 > t_1$ with $u(t_2) = 0$.

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Definition 1.2. **(Weak oscillation of $\mathcal{C}_0$–semigroups).** A strongly continuous semigroup T on a Banach space X *oscillates with respect to a subset* Φ of X' if $\langle \varphi, T(\cdot)x \rangle$ oscillates (in sense of Definition 1.1) for all $\varphi \in \Phi$ and all $x \in X$. We call T *weakly oscillating* if it oscillates with respect to X'.

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Proposition 2.1. (Necessary condition for weak oscillation). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup T on a Banach space X and let Φ be a subset of X' such that*

(1) Φ *separates the points of* X.

If T weakly oscillates with respect to Φ, then $P\sigma(A) \cap \mathbb{R} = \emptyset$.

$$\mathcal{O}(\Phi, A) := \left\{(\varphi, x) : \varphi \in \Phi, x \in X, \liminf_{\lambda \downarrow -\infty} e^{\alpha\lambda}\langle \varphi, R(\lambda, A)x\rangle = \infty \text{ for all } \alpha \in \mathbb{R}\right\}.$$

(**Theorem 2.2. Sufficient condition for weak oscillation).** *Let* $(A, D(A))$ *be the generator of a strongly continuous semigroup* T *on a Banach space* X *and let* Φ *be a subset of* X'. *If*

$$\sigma(A) \cap \mathbb{R} = \emptyset, \quad \textit{and if} \tag{3}$$

$$\langle \varphi, T(\cdot)x\rangle \ \textit{oscillates for each } (\varphi, x) \in \mathcal{O}(\Phi, A), \tag{4}$$

then T *weakly oscillates with respect to* Φ.

Journal of Differential Equations **180**, 171–197 (2002)
doi:10.1006/jdeq.2001.4051, available online at http://www.idealibrary.com on IDEAL®

Oscillation Theory of Linear Systems

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Received August 3, 2000

Definition 1.1. We say that a function $f : \mathbb{T}_+ \to \mathbb{R}$ *strongly oscillates* if for every $t \in \mathbb{T}_+$ such that $f(t) \neq 0$ there exists $s \in \mathbb{T}_+, s > t$ such that $f(s)\, f(t) < 0$.

Definition 1.2. A function $f : \mathbb{T}_+ \to X$ *strongly oscillates* if for every $\xi \in X^*$ the function $\xi \circ f : \mathbb{T}_+ \to \mathbb{R}$ strongly oscillates.

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wedge W in X a closed convex subset of X such that $\alpha W = W$

$$wedge(A) := cl\left\{\sum \alpha_i a_i \;\; | \, \alpha_i \in \mathbb{R}_+,\, a_i \in A\right\}.$$

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THEOREM 2.1. *Let $T:\mathbb{T} \times X \to X$ be a linear semidynamical system. Let $x \in X$. Then $x \in X$ strongly oscillates if and only if*

$$-x \in wedge(orb_T(x)). \tag{6}$$

Theorem 3.1. *Let $A \in \mathscr{L}(X)$ be such that*

$$\sigma(A) \cap \mathbb{R}_+ = \varnothing.$$

Then every point of X strongly oscillates in the discrete semidynamical system generated by A.

Corollary 3.1. *Let T be a uniformly continuous semigroup and let $A \in \mathscr{L}(X)$ be its generator. If $\sigma(A) \cap \mathbb{R} = \varnothing$ then every point in X strongly oscillates.*

THEOREM 4.1. *Let T be a strongly continuous semigroup and let A denote its generator. We assume that $\sigma(A) \cap \mathbb{R} = \varnothing$. Then there exists a residual subset S of X such that every point of S strongly oscillates.*

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THEOREM 4.2. *Let T be a strongly continuous group and let the generator A of T satisfy $\sigma(A) \cap \mathbb{R} = \emptyset$. Then every point of X strongly oscillates.*

DEFINITION 4.1. Let $f : \mathbb{R}_+ \to \mathbb{R}$. We say that f is *integrally small* if

$$\lim_{r \to \infty} \int_0^r f(t)\, e^{kt}\, dt \text{ exists for every } k \in \mathbb{R}.$$

THEOREM 4.3. *Let T be a strongly continuous semigroup and let A be the generator of T. We assume that*

$$\sigma(A) \cap \mathbb{R} = \varnothing.$$

Let $x \in X$ and $\xi \in X^*$ be such that $\xi(T(\cdot)\, x)$ is eventually nonnegative. Then x is ξ-integrally small.

$\xi \circ f$ is integrally small.

Portugal. Math. (N.S.)
Vol. 72, Fasc. 2-3, 2015, 193–206
DOI 10.4171/PM/1964

Portugaliae Mathematica

On the strong oscillatory behavior of all solutions to some second order evolution equations

Alain Haraux

(Communicated by Hugo Beirão da Veiga and José Francisco Rodrigues)

Definition 1.1. We say that a number $M > 0$ is a strong oscillation length for a numerical function $g \in L^1_{loc}(\mathbb{R})$ if the following alternative holds: either $g(t) = 0$ almost everywhere, or for any interval J with $|J| \geq M$, we have

$$\text{meas}\{t \in J, f(t) > 0\} > 0 \quad \text{and} \quad \text{meas}\{t \in J, f(t) < 0\} > 0.$$

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$$\text{meas}\{t \in J, f(t) > 0\} > 0 \quad \text{and} \quad \text{meas}\{t \in J, f(t) < 0\} > 0.$$

$$u'' + Au(t) = 0,$$

Proposition 1.2. *Under the above conditions on H, V and A, for any solution $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ of (1.1) and for any $\zeta \in V'$, the function $g(t) := \langle \zeta, u(t) \rangle$ has some finite strong oscillation number $M = M(u, \zeta)$.*

Proposition 2.1. *Let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be the sequence of eigenvalues of* A *repeated according to multiplicity and setting* $\mu_n := \{\lambda_n\}^{1/2}$, *assume that*

$$T = 2\pi \sum_n \frac{1}{\mu_n} < \infty$$

Then for any solution $u \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ *of* (1.1) *and any* $\zeta \in V'$, *the function* $g(t) := \langle \zeta, u(t) \rangle$ *has a strong oscillation length equal to* T.

Oscillation is robust under small perturbations

Thank you