Some results on spectral theory for suprema preserving operators on max-cones

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Aljoša Peperko, University of Ljubljana and IMFM Ljubljana;
(joint work with Vladimir Müller, Czech Academy of Sciences)

Talk mainly based on:

- V. Müller and A. Peperko, On the Bonsall cone spectral radius and the approximate point spectrum, Discrete and Continuous Dynamical Systems - Series A, 2017.

- V. Müller and A. Peperko, Lower spectral radius and spectral mapping theorem for suprema preserving mappings, Discrete and Continuous Dynamical Systems - Series A, 2018.

- V. Müller and A. Peperko, On some spectral theory for infinite bounded non-negative matrices in max algebra, LAMA, 2023, https://doi.org/10.1080/03081087.2023.2188155

X is a **normed vector lattice** (also called a Riesz space):

- X is an **ordered vector space** with a positive cone X_+;

- for every $x, y \in X$ there exist a **supremum** $x \vee y$ and an **infimum** $x \wedge y$ in X;

- $\| \cdot \|$ is a **lattice norm**: $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$. Note that $\||x|\| = \|x\|$ for $x \in X$.

  If $\| \cdot \|$ is complete, then X is called a **Banach lattice**.

- $C \subset X$ is called a **cone** if $tC = \{tx : x \in C\} \subset C$ for all $t \geq 0$
- a wedge is a convex cone
- a cone C is a **max-cone** if for each $x, y \in C$ we have $x \vee y \in C$

- $T : C \to C$ positively homogeneous and bounded, i.e., $\|T\| := \sup\{\|Tx\| : x \in C, \|x\| \leq 1\} < \infty$.

- $m(T) := \inf\{\|Tx\| : x \in C, \|x\| = 1\}$ minimum modulus of T

- $r(T) := \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n} \ldots$ **Bonsall's cone spectral radius** of T

- $d(T) := \lim_{n\to\infty} m(T^n)^{1/n} = \sup_n m(T^n)^{1/n}$ **lower cone spectral radius** of T

- **approximative point cone spectrum** $\sigma_{ap}(T)$ the set of all $s \geq 0$ such that $\inf\{\|Tx - sx\| : x \in C, \|x\| = 1\} = 0$

- **point cone spectrum**: $\sigma_p(T) = \{s \geq 0 : Tx = sx, x \in C, \|x\| = 1\}$

- local cone spectral radius at $x \in C$: $r_x(T) := \limsup_{n\to\infty} \|T^n x\|^{1/n} \leq r(T)$

Motivation, examples: our results apply to different **max type or min type operators** (Mallet-Parret, Nussbaum; Litvinov, Maslov; Akian,Gaubert, Walsh; Sturmfels; Butkovič; Heidergott, Olsder, de Woude; ...)

(Mallet-Parret, Nussbaum 03, 10) $T : C[0,a] \to C[0,a]$, $C = C_+[0,a]$, $TC \subset C$

$$(T(x))(s) = \max_{t \in [\alpha(s),\beta(s)]} k(s,t)x(t),$$

where $x \in C[0,a]$ and $\alpha, \beta : [0,a] \to [0,a]$ are given continuous functions satisfying $\alpha \leq \beta$.

The kernel $k : \mathcal{S} \to [0,\infty)$ is a given **non-negative continuous function**, where $\mathcal{S}$ denotes the compact set

$$\mathcal{S} = \{(s,t) \in [0,a] \times [0,a] : t \in [\alpha(s),\beta(s)]\}.$$

$T|_C : C \to C$ is a **positively homogeneous**, **Lipschitz** map that **preserves finite suprema**: $T(x \vee y) = Tx \vee Ty$, $x, y \in C$

The **eigenproblem** of these operators arises in the asymptotic (small ε) **study of slowly oscilating periodic solutions** of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t-\tau)), \quad \tau = \tau(y(t)),$$

with **state-dependent delay**

(Mallet-Parret, Nussbaum 2003, 2010), MP17 proved some **Krein-Rutman** type results, in particular:

If X a is Banach lattice, $C = X_+$, $T : C \to C$ positively homogeneous, continuous (and hence bounded), preserves finite suprema + **generalized compactness type conditions** on T (if a suitable version of essential radius is smaller than $r(T)$)

Then there exist $x \in C$, $x \neq 0$ such that $Tx = r(T)x$.

(MP 17,18) **Sup type operators on bounded functions** (special case: bounded matrices in max-algebra)

M nonempty set, X bounded real functions on M, norm $\|f\|_\infty = \sup\{|f(t)| : t \in M\}$ and natural operations, X is a normed vector lattice. Let $C = X_+$ and let $k : M \times M \to [0, \infty)$ satisfy $\sup\{k(t,s) : t, s \in M\} < \infty$. Let $T : C \to C$ be defined by $(Tf)(s) = \sup\{k(s,t)f(t) : t \in M\}$ and so $\|T\| = \sup\{k(t,s) : t, s \in M\}$. Clearly C is a max-cone, T is Lipschitz, positive homogeneous and preserves finite suprema.

The special case $M = \mathbb{N}$ (MP22+), (infinite bounded non-negative matrices $A = [a(i,j)] = [a_{ij}]$ i.e., $a(i,j) \geq 0$ for all $i,j \in \mathbb{N}$ and $\|A\|_\infty = \|a\|_\infty = \sup_{i,j\in\mathbb{N}} a(i,j) < \infty$,

$X = l^\infty$, $C = l^\infty_+$, $\|T\| = \|A\|_\infty$).

Associated infinite digraph: $G(A) = (V(A), E(A))$

$V(A) = \mathbb{N}$, $E(A) = \{(i,j) : a_{ij} > 0\}$.

Denote $T = T_A : C \to C$, $C = l_+^\infty$,

$(T_A x)(i) = (A * x)_i = \sup_{j \in \mathbb{N}} a(i,j) x_j, \ i \in \mathbb{N}, x \in l_+^\infty$;

$(T_A T_B x)(i) = \sup_{j,k \in \mathbb{N}} a(i,k) b(k,j) x_j$ corresponds to a product matrix in max-algebra $(A * B)_{ij} = \sup_{k \in \mathbb{N}} a(i,k) b(k,j)$.

In the case $M = \{1, ..., n\}$ for a given $n \in \mathbb{N}$, $X = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ the topic is well studied and known under the name **max algebra** (an isomorphic version of **tropical** max-plus or min -plus algebra) with diverse (semi)field of applications including problems from:

machine-scheduling, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, information technology, DNA analysis, graph theory, ...

The usefulness of this setting arises from a fact that some clasically **non-linear problems** can be described and studied in a **linear fashion**

What can be proved without any generalized compactness assumptions?

Theorem 1 *(MP 17) Let X be a normed vector lattice, let $C \subset X_+$ be a non-zero max-cone. Let $T : C \to C$ be a mapping which is bounded, positively homogeneous and preserves finite suprema. Let $C' \subset C$ be a bounded subset satisfying $\|T^n\| = \sup\{\|T^n x\| : x \in C'\}$ for all n. Then*

$$[\sup\{r_x(T) : x \in C'\}, r(T)] \subset \sigma_{ap}(T).$$

In particular, $r(T) \in \sigma_{ap}(T)$. Moreover, $r_x(T) \in \sigma_{ap}(T)$ for each $x \in C$, $x \neq 0$.

If, in addition, T is a Lipschitz, then $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$.

Proof is quite technical, important ingredient:

Lemma 2 *Let* X *be a normed vector lattice and let* $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. *Then*

$$\| \bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \| \leq \sum_{j=1}^{n} \|x_j - y_j\|.$$

Proof: Birkhoff inequality

$$| \bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j | \leq \sum_{j=1}^{n} |x_j - y_j|$$

and **lattice property** of the norm.

Proposition 3 *Let X be a normed space and $C \subset X$ a non-zero cone. If $T : C \to C$ is positively homogeneous and bounded, then*

$$d(T) \leq \inf\{r_x(T) : x \in C, x \neq 0\} \leq \sup\{r_x(T) : x \in C, x \neq 0\} \leq r(T). \tag{1}$$

If, in addition, T is Lipschitz, then $\sigma_{ap}(T) \subset [d(T), r(T)]$.

Analogue of the previous theorem for $d(T)$?

Remark. $\sigma_{ap}(T)$ may not contain the whole interval $[d(T), \inf\{r_x(T) : x \in C,\ x \neq 0\}]$!

Example 4 Let $X = \ell^\infty$ with the standard basis $e_{n,k} \quad (n,k \in \mathbb{N})$. More precisely, the elements of X are formal sums $x = \sum_{n,k\in\mathbb{N}} \alpha_{n,k} e_{n,k}$ with real coefficient $\alpha_{n,k}$ such that

$$\|x\| := \sup\{|\alpha_{n,k}| : n,k \in \mathbb{N}\} < \infty.$$

Then X is a Banach lattice with the natural order. Let $C = X_+$ and let $T : C \to C$ be defined by $Te_{n,1} = n^{-1}e_{n,2}$, $Te_{n,k} = e_{n,k+1} \quad (k \geq 2)$. More precisely,

$$T(\sum_{n,k\in\mathbb{N}} \alpha_{n,k}e_{n,k}) = \sum_{n\in\mathbb{N}}(\alpha_{n,1}n^{-1}e_{n,2} + \sum_{k=2}^{\infty} \alpha_{n,k}e_{n,k+1}).$$

Then T is positively homogeneous, additive, Lipschitz mapping that preserves finite suprema, such that $d(T) = 0$ and $r_x(T) = 1$ for all non-zero $x \in C$. Moreover, $\sigma_{ap}(T) = \{0,1\}$ and so $\sigma_{ap}(T)$ does not contain the whole interval

$$[d(T), \inf\{r_x(T) : x \in C,\ x \neq 0\}].$$

Theorem 5 *(MP18) Let X be a normed vector lattice, let $C \subset X_+$ be a non-zero max-cone. Let $T : C \to C$ be a mapping which is bounded, positively homogeneous and preserves finite suprema. Then $d(T) \in \sigma_{ap}(T)$.*

If, in addition, T is Lipschitz, then $d(T) = \min\{t : t \in \sigma_{ap}(T)\}$.

Theorem 6 *(MP17) Let X be a* **normed space**, $C \subset X$ *a non-zero* **normal wedge** *and let $T : C \to C$ be positively homogeneous,* **additive** *and Lipschitz. Let $C' \subset C$ be a bounded subset satisfying $\|T^n\| = \sup\{\|T^n x\| : x \in C'\}$ for all n. Then*

$$[\sup\{r_x(T) : x \in C'\}, r(T)] \subset \sigma_{ap}(T).$$

In particular, $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$.

Moreover, $r_x(T) \in \sigma_{ap}(T)$ for each $x \in C$, $x \neq 0$.

Theorem 7 *(MP18) Let X be a normed space, $C \subset X$ a non-zero normal wedge and let $T : C \to C$ be positively homogeneous, additive and bounded. Then $d(T) \in \sigma_{ap}(T)$.*

If, in addition, T is Lipschitz, then $d(T) = \min\{t : t \in \sigma_{ap}(T)\}$.

Some additional results:

- (MP 18) polynomial spectral mapping theorem for σ_{ap} for positively homogeneous, Lipschitz mappings that preserve finite suprema (but not for σ_p in general!)

- (MP 22+) upper semicontinuity for the maps $T \mapsto r(T), T \mapsto \sigma_{ap}(T)$ (not continuous in general!; continuous under suitable assumptions)

By Theorem 1 the following result follows ($C' = \{e_j : j \in \mathbb{N}\}$).

Corollary 8 *Let A be an infinite bounded non-negative matrix and let* $\sup\{r_{e_j}(T_A) : j \in \mathbb{N}\} \leq t \leq r(T_A)$. *Then $t \in \sigma_{ap}(T_A)$.*

(Recall that $(T_A x)(i) = \sup_{j\in\mathbb{N}} a(i,j)x_j,\ i \in \mathbb{N}, x \in l^\infty_+$;)

It may happen that $\sup\{r_{e_j}(T_A) : j \in \mathbb{N}\} \neq r(T_A)$.

Example. Backward shift $T_A : C \to C$, $C = l^\infty_+$,
$T_A(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$, i.e., $T_A e_1 = 0$ and $T_A e_j = e_{j-1}$ for all $j \geq 2$. Then $r_{e_j}(T_A) = 0$ for each element e_j, but $r(T_A) = 1$. We also have $\sigma_{ap}(T_A) = [0, 1] = \sigma_p(T_A)$.

On the other hand, for its restriction $T_A|_{c_0^+}$, to the positive cone of the space of null convergent sequences c_0^+, we have $r(T_A|_{c_0^+}) = 1$, $\sigma_p(T_A|_{c_0^+}) = [0, 1)$ and $\sigma_{ap}(T_A|_{c_0^+}) = [0, 1]$.

For $i_0, i_1, i_2, \ldots, i_k \in \mathbb{N}$ write for short

$A(i_k, \ldots, i_0) = a(i_k, i_{k-1}) \cdots a(i_2, i_1) a(i_1, i_0)$ and so

$\|T_A^k\| = \sup\{A(i_k, \ldots, i_0) : i_0, \ldots, i_k \in \mathbb{N}\}$. Define

$$\mu(A) = \sup\{(A(i_1, i_k, \ldots, i_2, i_1)^{1/k} : k \in \mathbb{N}, i_1, \ldots, i_k \in \mathbb{N}\}.$$

One can assume that the vertices $i_1, \ldots, i_k$ in the definition of $\mu(A)$ are mutually distinct.
For $k \in \mathbb{N}$ write $c_k(A) = \sup\{A(i_k, \ldots, i_0) : i_0, \ldots, i_k \in \mathbb{N}$ mutually distinct$\}$ and

$$r'(A) = \limsup_{k \to \infty} c_k(A)^{1/k}.$$

Theorem 9 *If A is a non-negative bounded matrix then*

$$r(T_A) = \max\{\mu(A), r'(A)\}.$$

Define $s(T_A) = \sup_j r_{e_j}(T_A)$ and $s_e(T_A) = \limsup_{j\to\infty} r_{e_j}(T_A)$.

For $n \in \mathbb{N}$ let $P_n : \ell^\infty_+ \to \ell^\infty_+$ be the canonical projection defined by $P_n(x_1, x_2, \ldots) = (\underbrace{0, \ldots, 0}_{n}, x_{n+1}, \ldots)$.

Let $r_{ess}(T_A) = \lim_{n\to\infty} r(P_n T_A P_n) = \inf_{n\in\mathbb{N}} r(P_n T_A P_n)$.

Then $s_e(T_A) \le s(T_A) \le r(T_A)$ and $r_{ess}(T_A) \le r(T_A)$.

Theorem 10 $\mu(A) \le s(T_A)$ *and* $r'(A) \le r_{ess}(T_A)$. *Consequently,*

$$r(T_A) = \max\{r_{ess}(T_A), s(T_A)\}.$$

Theorem 11 *Let* $A \in \mathbb{R}_+^{\infty \times \infty}$ *and* $r_{ess}(T_A) < r(T_A)$. *Then* $r(T_A) \in \sigma_p(T_A)$.

The assumption $r_{ess}(T_A) < r(T_A)$ is necessary for the conclusion of Theorem 11 as the following example shows.

Example 12 Let $a_{i,i-1} = 1$ for all $i \in \mathbb{N}$, $i \geq 2$ and $a_{i,j} = 0$ otherwise (A is a forward shift). Then $r(T_A) = r_{ess}(T_A) = r'(A) = s(T_A) = s_e(T_A) = 1$, $\mu(A) = 0$ and 1 is not in $\sigma_p(T_A) = \emptyset$.

Theorem 13 *Let a nonnegative bounded matrix A satisfy $s_e(T_A) < s(T_A)$. Then there exists a sequence (finite or infinite) of finite nonempty disjoint sets $F_1, F_2, \ldots \subset \mathbb{N}$ and numbers $s(T_A) = s_1 > s_2 > s_3 \cdots$ such that in the decomposition $\mathbb{N} = F_1 \cup F_2 \cup \cdots \cup (\mathbb{N} \setminus \bigcup F_j)$, the matrix A is permutationally equivalent to a matrix in the form*

$$A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots \\ * & A_{22} & 0 & \cdots \\ * & * & A_{33} & \cdots \\ \vdots & & & A_{\infty,\infty} \end{pmatrix}$$

where $r_{e_j}(T_A) = s(T_{A_{kk}}) = s_k$ for all $j \in F_k$.

If the sequence (s_k) is finite then $s_e(T_A) = s(T_{A_{\infty,\infty}})$.

If the sequence (s_k) is infinite then $s_e(T_A) = \lim_{k\to\infty} s_k\}$.

If, in addition, $r_{ess}(T_A) < r(T_A)$ then there exists a decomposition with the above properties such that

$$r(T_{A_{kk}}) = \mu(A_{kk}) = s_k$$

for all k that satisfy $s_k > r_{ess}(T_A)$. Moreover, for such k the supremum (maximum) in the definition of $\mu(A_{kk})$ is attained.