FROM LOEWNER'S CHARACTERIZATION OF OPERATOR MONOTONE FUNCTIONS TO FUNDAMENTAL TH. OF CHRONOGEOMETRY

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H Hilbert space, $S(H)$ the set of all linear bounded self-adjoint operators on H

The usual partial order on $S(H)$:

$$A \leq B \iff$$
$$\langle Ax, x \rangle \leq \langle Bx, x \rangle \text{ for every } x \in H$$

The finite-dimensional case:

H_n the set of all $n \times n$ hermitian matrices

$$A = U \begin{bmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_n \end{bmatrix} U^*$$

$$A \geq 0 \iff$$

all eigenvalues of A are non-negative.

$$A \leq B \iff B - A \geq 0$$

Molnár's theorem:

THEOREM $\phi : H_n \to H_n$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exist an invertible matrix T and $B \in H_n$ such that either

$$\phi(A) = TAT^* + B$$

for every $A \in H_n$, or

$$\phi(A) = TA^{tr}T^* + B$$

for every $A \in H_n$.

$$M = \{(x, y, z, t) \,:\, x, y, z, t \in \mathbf{R}\}$$

$$(x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) \in M \quad \text{coherent}$$

$$\Updownarrow$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = c^2(t_1 - t_2)^2$$

In mathematical foundations of relativity we usually use the harmless normalization $c = 1$.

Two space-time events are coherent (light-like) $\iff$ a light signal can be sent from one to the other

Alexandrov: description of bijective maps on M preserving coherency in both directions

$$r = (x, y, z, t) \leftrightarrow \begin{bmatrix} t + z & x + iy \\ x - iy & t - z \end{bmatrix} = A$$

$$A \in H_2$$

$$\det A = t^2 - z^2 - x^2 - y^2$$

$$r_1, r_2 \in M, \quad r_j \leftrightarrow A_j$$

$$r_1, r_2 \ \text{ coherent } \iff \det(A_2 - A_1) = 0$$

$$\Updownarrow$$

$$A_2 - A_1 \ \text{ singular}$$

$$\Updownarrow$$

$$A_1 = A_2 \ \text{ or } \quad A_1 \text{ and } A_2 \text{ adjacent}$$

$$A_1, A_2 \text{ adjacent} \iff \text{rank}\,(A_1 - A_2) = 1$$

Thus, Alexandrov problem = study of adjacency preservers on H_2

$A, B \in H_n$, $A \neq B$. TFAE:

- A, B adj.
- A, B comparable and if C, D belong to operator interval between A and B, then C and D comparable.

Proof. ($\Downarrow$)
$B = A + tP$, say $t > 0 \Rightarrow A \leq B$

$$[A, B] = \{A + sP \,:\, 0 \leq s \leq t\}$$

$C, D \in [A, B] \Rightarrow C = A + s_1 P, D = A + s_2 P.$

($\Uparrow$) A, B not adjacent
If A, B not comparable, done.
If comparable, WLOG $A \leq B$. rank $(B - A) \geq 2 \Rightarrow$ "enough room" to find two noncomparable in $[A, B]$.

Effect algebra E_n:

$$E_n = \{A \in H_n \,:\, 0 \le A \le I\}$$

Orthocomplementation on E_n:

$$A \in E_n \,:\quad A^\perp = I - A$$

THEOREM (Ludwig, characterization of ortho-order automorphisms of E_n).
$\phi : E_n \to E_n$ a bijective map such that

$$A \le B \iff \phi(A) \le \phi(B)$$

and

$$\phi(A^\perp) = \phi(A)^\perp.$$

Then there exists a unitary matrix U such that either

$$\phi(A) = UAU^*$$

for every $A \in E_n$, or

$$\phi(A) = UA^{tr}U^*$$

for every $A \in E_n$.

Molnár: bijectivity + order preserving

Ludwig: bijectivity + order preserving + orthocomplementation preserving

CONJECTURE. $\phi : E_n \to E_n$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exists a unitary matrix U such that either

$$\phi(A) = UAU^*$$

for every $A \in E_n$, or

$$\phi(A) = UA^{tr}U^*$$

for every $A \in E_n$.

Wrong!

p a real number, $p < 1$.

$$f_p : [0,1] \to [0,1]$$

$$f_p(x) = \frac{x}{px + (1-p)}, \quad x \in [0,1].$$

THEOREM. $n \geq 2$. $\phi : E_n \to E_n$ bijective.

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\Downarrow$$

$\exists p, q \in (-\infty, 1)$, $\exists$ an invertible matrix T with $\|T\| \leq 1$ such that either

$$\phi(A) =$$

$$= f_q \left((f_p(TT^*))^{-1/2} \, f_p(TAT^*) \, (f_p(TT^*))^{-1/2} \right),$$

or

$$\phi(A) =$$

$$= f_q \left((f_p(TT^*))^{-1/2} \, f_p(TA^{tr}T^*) \, (f_p(TT^*))^{-1/2} \right).$$

Next problem?

$A, B \in H_n,\ A < B.$

$$[A, B] = \{C \in H_n \,:\, A \le C \le B\},$$

$$[A, B) = \{C \in H_n \,:\, A \le C < B\},$$

$$(A, B) = \{C \in H_n \,:\, A < C < B\}.$$

$$[A, \infty) = \{C \in H_n \,:\, C \ge A\},$$

$$(A, \infty) = \{C \in H_n \,:\, C > A\},$$

$$(-\infty, \infty) = H_n$$

$$(A, B],\ (-\infty, A],\ (-\infty, A)$$

Which of the above operator intervals are order isomorphic?

The general form of all order isomorphisms between operator intervals that are order isomorphic?

Each operator interval J is isomorphic to one of the following operator intervals:

$[0, I]$
$[0, \infty)$
$(-\infty, 0]$
$(0, \infty)$
$(-\infty, \infty)$

And any two of these operator intervals are order non-isomorphic.

The operator intervals $[0, \infty)$ and $(-\infty, 0]$ are obviously order anti-isomorphic. Hence, to understand the structure of all order isomorphisms between any two order isomorphic operator intervals it is enough to describe the general form of order automorphisms of the following four operator intervals:

$[0, I]$
$[0, \infty)$
$(0, \infty)$
$(-\infty, \infty)$

The group of order automorphisms of $[0, I]$ and $(-\infty, \infty)$: previous slides

THEOREM $\phi : [0, \infty) \to [0, \infty)$ a bijective map such that

$$A \le B \iff \phi(A) \le \phi(B).$$

Then there exists an invertible matrix T such that either

$$\phi(A) = TAT^*$$

for every $A \in [0, \infty)$, or

$$\phi(A) = TA^{tr}T^*$$

for every $A \in [0, \infty)$.

THEOREM $\phi : (0, \infty) \to (0, \infty)$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exists an invertible matrix T such that either

$$\phi(A) = TAT^*$$

for every $A \in (0, \infty)$, or

$$\phi(A) = TA^{tr}T^*$$

for every $A \in (0, \infty)$.

Groups of automorphisms:

- $[0, \infty), (0, \infty)$: simple.
- $(-\infty, \infty)$: simple.
- $[0, I]$ more complicated? NO.

Questions:

$$0 \leq A \leq B \Rightarrow A^2 \leq B^2 \ ?$$

$$0 \leq A \leq B \Rightarrow A^{1/2} \leq B^{1/2} \ ?$$

$f : (a, b) \to \mathbb{R}$ operator monotone if for all n

$$\forall A, B \in H_n, \;\; \sigma(A), \sigma(B) \subset (a, b) :$$

$$A \leq B \Rightarrow f(A) \leq f(B)$$

TH (Loewner). $f : (a, b) \to \mathbb{R}$ operator monotone $\iff$ f has an analytic continuation to the upper half-plane Π which maps Π into itself.

B. Simon, Loewner's theorem on monotone matrix functions, Grundlehren Math. Wissen. **354**, Springer, 2019.

Thus,

$$0 \leq A \leq B \not\Rightarrow A^2 \leq B^2$$

$$0 \leq A \leq B \Rightarrow A^{1/2} \leq B^{1/2}$$

THEOREM. Let $U \subset S(H)$ be an operator domain. The following conditions are equivalent for a map $\phi : U \to S(H)$.

- The map ϕ is a local order isomorphism.
- The map ϕ has a unique continuous extension to $U \cup \Pi(H)$ that maps $\Pi(H)$ biholomorphically onto itself.

operator domain
local order isomorphism
$\Pi(H)$

Uniqness principle

Maximal loc ord isomorphisms

Equivalence relation: $\phi = \xi_1 \circ \psi \circ \xi_2$

$\hat{U}_A := \{X \in S(H) \;:\; XA{+}I \;\text{ is invertible}\}$

$0 \in U_A$, the connected component of $\hat{U}_A$ in $S(H)$.

$$\Phi_A : U_A \to S(H)$$
$$\Phi_A(X) = (XA + I)^{-1}X.$$

Φ_A bijection of U_A onto U_{-A}, max loc ord isomorphism

Each maximal loc ord isomorphism equivalent to some Φ_A.

Finite-dimensional case:

$$U(m,p) = \{X \in H_n \,:\, X_{11} \in H_m(p)\}.$$

Clearly, $U(m,p)$ is a matrix domain in H_n. Define a map $\phi_{m,p} : U(m,p) \to U(m,m-p)$:
For

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \in U(m,p)$$

$$\phi_{m,p}(X) =$$
$$\begin{bmatrix} -X_{11}^{-1} & iX_{11}^{-1}X_{12} \\ -iX_{12}^*X_{11}^{-1} & X_{22} - X_{12}^*X_{11}^{-1}X_{12} \end{bmatrix}$$

$$
- \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{12}^* & X_{22} & iI \\ 0 & -iI & 0 \end{bmatrix}^{-1}
$$

$$
= \begin{bmatrix} -X_{11}^{-1} & 0 & iX_{11}^{-1}X_{12} \\ 0 & 0 & -iI \\ -iX_{12}^*X_{11}^{-1} & iI & X_{22} - X_{12}^*X_{11}^{-1}X_{12} \end{bmatrix}
$$

$$X \to X$$

$$X \to -X^{-1}$$