Recent progress in Banach lattices

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Based on recent collaborations with A. Avilés, D. de Hevia, G. Martínez-Cervantes, T. Oikhberg, J. Rodríguez, A. Rueda Zoca, M. A. Taylor, V. G. Troitsky...

Banach lattice = Banach space + vector lattice + $\left[|x| \leq |y| \Rightarrow \|x\| \leq \|y\|\right]$

T lattice homomorphism: T linear + $|Tx| = T|x|$

Banach space properties	**Banach lattice properties**
Reflexive	No c_0, ℓ_1 sublattices
Weakly sequentially complete	No c_0 sublattices
Type / cotype	Convexity / concavity

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Lattice vs. linear embeddings

Theorem (Meyer-Nieberg 1973)

c_0 has a linear embedding in a Banach lattice $\Leftrightarrow$ c_0 embeds as a sublattice.

Theorem (Avilés-MartínezCervantes-RuedaZoca-T 2022)

$C[0,1]$ has a linear embedding in a Banach lattice $\Leftrightarrow$ $C[0,1]$ embeds as a sublattice.

Theorem (Avilés-MartínezCervantes-RuedaZoca-T 2022)

X separable Banach lattice. TFAE:

1. *X has a lattice embedding in a Banach lattice whenever there is a linear embedding.*
2. *X is a sublattice of $C[0,1]$.*

Question: non-separable case?

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Complemented subspaces

Question: Suppose E is a complemented subspace of a Banach lattice. Is E isomorphic to some Banach lattice?

Related questions: complemented subspaces of L_1 or $C(K)$?

Theorem (Abramovich-Wojtaszyk 1975)

$\mathcal{L}_1$ space isomorphic to a Banach lattice must be an $L_1(\mu)$ space.
$\mathcal{L}_\infty$ space isomorphic to a Banach lattice must be a sublattice of some $C(K)$ space.

Theorem (Benyamini 1973)

Every separable sublattice of $C(K)$ is isomorphic to some $C(L)$.

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Theorem (Plebanek-Salguero 2022)

There exist a (non-separable) space PS_2 which is 1-complemented in certain $C(K)$ but not isomorphic to any $C(L)$.

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PS_2 is not isomorphic to any Banach lattice.

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Let E be a Banach space. $FBL[E]$ is a Banach lattice with $\delta : E \to FBL[E]$ linear isometry,

$\forall$ X Banach lattice and $T : E \to X$ $\exists!$ lattice homomorphism $\hat{T}$

moreover, $\|\hat{T}\| = \|T\|$.

- $FBL[\ell_1(A)]$ for any set A. [de Pagter-Wickstead, 2015]
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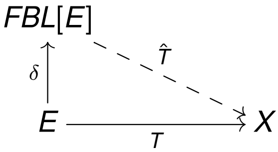

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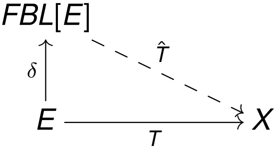

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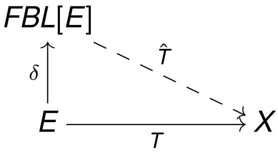

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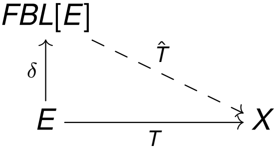

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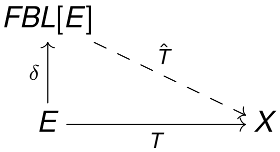

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$$\begin{array}{ccc} FBL[E] & & \\ \uparrow \delta & \overset{\hat{T}}{\searrow} & \\ E & \xrightarrow[T]{} & X \end{array}$$

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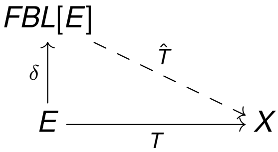

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Every linear operator $T : E \to F$ between Banach spaces extends uniquely to a lattice homomorphism $\overline{T}$ as follows

$$\begin{array}{ccc} FBL[E] & \overset{\overline{T}}{\dashrightarrow} & FBL[F] \\ \uparrow{\scriptstyle \delta_E} & & \uparrow{\scriptstyle \delta_F} \\ E & \underset{T}{\longrightarrow} & F \end{array}$$

Theorem (Oikhberg-Taylor-T-Troitsky)

1. *T is injective iff $\overline{T}$ is injective.*
2. *T is surjective iff $\overline{T}$ is surjective.*

Theorem (OTTT)

Suppose $\iota : E \hookrightarrow F$. $\bar{\iota}(FBL[E])$ is a sublattice of $FBL[F]$ $\Leftrightarrow$ every operator $S : E \to \ell_1^n$ admits a norm-preserving extension $\tilde{S} : F \to \ell_1^n$.

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Dreaming a metatheorem

E has Banach space property (P) if and only if $FBL[E]$ has Banach lattice property (P')

Theorem

E is finite dimensional iff $FBL[E]$ has a strong unit.

Theorem

E is separable iff $FBL[E]$ has a quasi-interior point.

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If E is WCG, then $FBL[E]$ is LWCG.

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For a Banach space E, TFAE:

1. *E contains a complemented subspace isomorphic to ℓ_1.*
2. *$FBL[E]$ contains a lattice complemented sublattice isomorphic to ℓ_1.*
3. *$FBL[E]$ contains a lattice complemented sublattice isomorphic to $FBL[\ell_1]$.*
4. *$FBL[E]$ contains a sublattice isomorphic to ℓ_1.*
5. *$FBL[E]$ has a lattice quotient isomorphic to ℓ_1.*

Note: $C[0,1]$ contains a subspace isomorphic to ℓ_1 but $FBL[C[0,1]]$ does not contain **any** sublattice isomorphic to ℓ_1.

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$$\Big\|\sum_k x_k\Big\| \leq C\Big(\sum_k \|x_k\|^p\Big)^{\frac{1}{p}}.$$

Recall: An operator between Banach spaces $T : E \to F$ is $(q, 1)$-summing if

$$\Big(\sum_k \|x_k\|^q\Big)^{\frac{1}{q}} \leq \pi_{(q,1)}(T) \sup_{\varepsilon_k=\pm 1} \Big\|\sum_k \varepsilon_k x_k\Big\|.$$

Theorem (OTTT)

Let E be a Banach space and $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. TFAE:

1. *id_{E^*} is $(q, 1)$-summing*
2. *$FBL[E]$ satisfies an upper p-estimate*
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Thank you for your attention!

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