Direct and Inverse Limits of Vector Lattices

Jan Harm van der Walt

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, South Africa

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Theorem

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(iii) $C(K)$ *has a separating order continuous dual.*

(iv) *Let $\mathcal{F}$ be a maximal singular family of order continuous functionals on $C(K)$, and for each $\varphi \in \mathcal{F}$ let C_φ denote its carrier and P_φ the band projection onto C_φ. Then*

$$C(K) \ni u \longmapsto (P_\varphi u)_{\varphi \in \mathcal{F}} \in \bigoplus_\infty C_\varphi$$

is an isometric lattice isomorphism.

The Bidual of $\mathrm{C}(K)$

What does the bidual of $\mathrm{C}(K)$ look like?

The Bidual of $\mathrm{C}(K)$

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Let K be a compact Hausdorff space. There exists a unique compact Hausdorff space $\tilde{K}$ such that the bidual $\mathrm{C}(K)^{**}$ of $\mathrm{C}(K)$ is isometrically lattice isomorphic to $\mathrm{C}(\tilde{K})$.

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Kakutani 3: $C(K)^{**}$ is a unital AM-space $\Rightarrow C(K)^{**} \cong C(\tilde{K})$ for some compact Hausdroff $\tilde{K}$

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- *Compact Hausdorff space $K \longrightarrow$ Realcompact space X.*
- *Norm (bi)dual of $C(K) \longrightarrow$ order (bi)dual of $C(X)$.*

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$$\varphi(u) = \int_X u \, d\mu, \ \ u \in C(X).$$

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The realcompactification

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Let X be a Tychonoff space. There exists a unique realcompact space υX so that $\mathrm{C}(X)$ and $\mathrm{C}(\upsilon X)$ are lattice and ring isomorphic.

The order dual of $C(X)$

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The order dual of $\mathrm{C}(X)$

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Let X be a realcompact space.

- $\mathrm{C}(X)^{\sim} = \mathrm{M}_c(X)$ *(compactly supported regular Borel measures on X).*

When is $\mathrm{C}(X)$ an order dual space?

Theorem (Xiong, 1983)

Let X be an extremally disconnected realcompact space, and let $S = \bigcup\{S_\varphi \ : \ 0 \leq \varphi \in \mathrm{C}(X)^\sim_\mathrm{n}\}$. TFAE:

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- *…?*

$$C(S) = \ldots?$$

Direct Limits

Categories of Vector Lattices

	Objects	Morphisms
VL	Vector lattices	Lattice homomorphisms
NVL	Vector lattices	Normal lattice homomorphisms
IVL	Vector lattices	Interval preserving lattice homomorphisms
NIVL	Vector lattices	Normal, interval preserving lattice homomorphisms
TOP	Topological spaces	Continuous functions

Definitions

Definition

Let **C** *be a category of vector lattices, I a directed set,* E_α *a vector lattice for each* $\alpha \in I$*, and* $e_{\alpha,\beta} : \mathrm{E}_\alpha \to \mathrm{E}_\beta$ *a* **C***-morphism for all* $\alpha \preccurlyeq \beta$ *in I.*

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- $\mathcal{D} := \left((\mathrm{E}_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta} \right)$ *is a direct system in* **C** *if, for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I,*

$$e_{\beta,\gamma} \circ e_{\alpha,\beta} = e_{\alpha,\gamma}.$$

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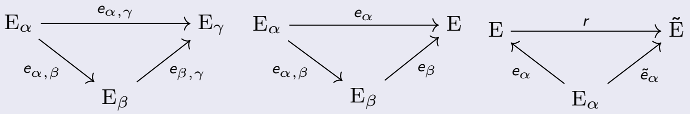

Examples

Example

- $n \le m$ in $\mathbb{N}$: $e_{n,m} : \mathbb{R}^n \ni (x_1, \dots x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0) \in \mathbb{R}^m$

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- $\varinjlim \mathbb{R}^n = c_{00}$ in **NIVL**.

Inverse Limits

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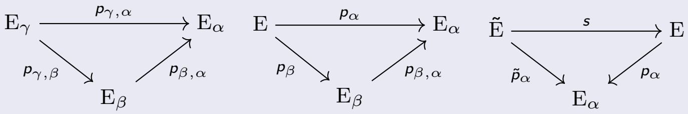

Examples

Example

- $n \leq m$ in $\mathbb{N}$: $p_{m,n} : \mathbb{R}^n \ni (x_1, \ldots x_m) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$

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- $\varprojlim \mathbb{R}^n = \mathbb{R}^\omega$ in **NVL**.

Duality

Order Adjoints

Theorem

$T : \mathrm{E} \to \mathrm{F}$ *a positive operator;* $T^{\sim} : \mathrm{F}^{\sim} \to \mathrm{E}^{\sim}$ *its order adjoint,* $\varphi \mapsto \varphi \circ T$.

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(iii) *T interval preserving $\Rightarrow$ $T^\sim$ a lattice homomorphism.*

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(iv) *T a lattice homomorphism $\Rightarrow T^{\sim}$ interval preserving.*

Order Adjoints

Theorem

$T : \mathrm{E} \to \mathrm{F}$ *a positive operator;* $T^{\sim} : \mathrm{F}^{\sim} \to \mathrm{E}^{\sim}$ *its order adjoint,* $\varphi \mapsto \varphi \circ T$.

(i) $T^{\sim}$ *is positive and order continuous.*

(ii) T *order continuous* $\Rightarrow$ $T^{\sim}[\mathrm{F}^{\sim}_{\mathrm{n}}] \subseteq \mathrm{E}^{\sim}_{\mathrm{n}}$.

(iii) T *interval preserving* $\Rightarrow$ $T^{\sim}$ *a lattice homomorphism.*

(iv) T *a lattice homomorphism* $\Rightarrow$ $T^{\sim}$ *interval preserving. The converse is true if* ${}^{\circ}\mathrm{F}^{\sim} = \{0\}$.

Dual Systems of Direct Systems

Proposition (VV)

(i) $\mathcal{D} := \left((\mathrm{E}_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta}\right)$ *a direct system in* **IVL** $\Rightarrow$
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Let $\mathcal{D} := \big((\mathrm{E}_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta} \big)$ *be a direct system in* **IVL**, *and* $\varinjlim \mathrm{E}_\alpha = \mathrm{E}$ *in* **IVL**. *Then* $\varprojlim {\mathrm{E}_\alpha}^{\sim} = \mathrm{E}^{\sim}$ *in* **NVL**.

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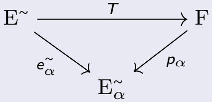

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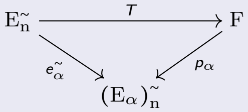

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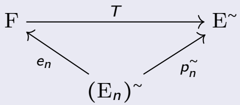

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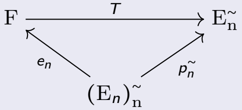

Inverse Limits of Perfect Spaces

Theorem (VV)

Let $\mathcal{I} \coloneqq \left((\mathrm{E}_\alpha)_{\alpha \in \mathcal{I}}, ({}_{\beta,\alpha})_{\beta \succcurlyeq \alpha} \right)$ *be an inverse system in* **NIVL**. *Assume that*

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Then E *is perfect.*

Solutions

When is $\mathrm{C}(X)$ an order dual space?

Theorem (Xiong, 1983)

Let X be an extremally disconnected realcompact space, and let $S = \bigcup\{S_\varphi \ : \ 0 \leq \varphi \in \mathrm{C}(X)^\sim_\mathrm{n}\}$. TFAE

- *There exists a vector lattice E so that $\mathrm{E}^\sim$ is lattice isomorphic to $\mathrm{C}(X)$.*
- $\mathrm{C}(X)$ *is perfect.*
- $\upsilon S = X$, *i.e.* $\mathrm{C}(X)$ *is lattice isomorphic to* $\mathrm{C}(S)$.
- *. . . ?*

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- *$\varprojlim \mathrm{C}_\varphi = \mathrm{C}(X)$ in* **NVL**, *with band projections as linking maps.*

The order bidual of $\mathrm{C}(X)$

The order bidual of $C(X)$

Example (DV, 2022)

X a realcompact space; $\mathcal{K} = \{K_\alpha \ : \ \alpha \in I\}$ the nonempty compact subsets of X.

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The order bidual of $C(X)$

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The order bidual of $\mathrm{C}(X)$

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- $e^{\sim}_{\alpha,\beta}(\mathbf{1}_{\tilde{K}_\beta}) = \mathbf{1}_{\tilde{K}_\alpha}$.
- There exists $\theta_{\alpha,\beta} : \tilde{K}_\alpha \to \tilde{K}_\beta$ continuous s.t. $e^{\sim}_{\alpha,\beta}(u) = u \circ \theta_{\alpha,\beta},\ u \in \mathrm{C}(\tilde{K}_\beta)$.

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The order bidual of $\mathrm{C}(X)$

Theorem (DV, 2022)

X a realcompact space. There exists a unique realcompact, extremally disconnected space Z so that $\mathrm{C}(X)^{\sim\sim}$ is lattice isomorphic to $\mathrm{C}(Z)$

The End