Continuity and robustness of risk measures

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Positivity XI

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This talk is based on a joint project with N. Gao and C. Munari

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- Let $\rho : \mathcal{X} \to \mathbb{R}$ be a law-invariant functional. The associated statistical functional $\mathcal{R}_\rho : \mathcal{M}(\mathcal{X}) \to \mathbb{R}$ is given by the formula

$$\mathcal{R}_\rho(\mathbb{P} \circ X^{-1}) := \rho(X).$$

for all $X \in \mathcal{X}$.

Statistical consistency

Let (X_n) be a sequence of independent identical random variables in L^0 with common law μ_0. Then we denote by

$$\widehat{m}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$

the empirical distribution of (X_n) and by

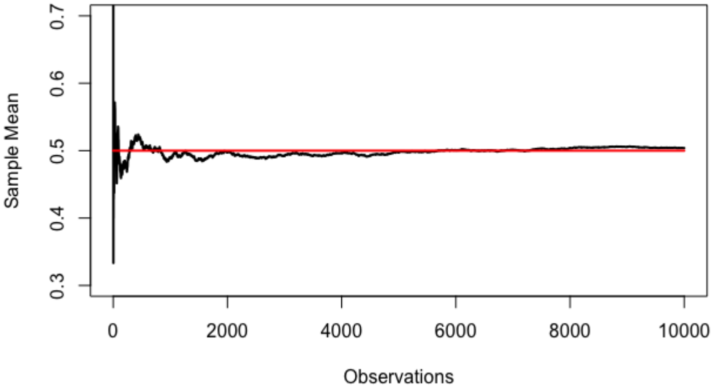
$$\widehat{\rho}_n := \mathcal{R}_\rho(\widehat{m}_n)$$

the corresponding estimate.
We say that ρ is statistical consistent at $X \in \mathcal{X}$ whenever

$$\widehat{\rho}_n = \mathcal{R}_\rho(\hat{m}_n) \xrightarrow{a.s.} \mathcal{R}_\rho(\mu_0) = \rho(X).$$

Law of large numbers

$$\mathcal{X} = L^1(\mathbb{P}), \rho(X) = \mathbb{E}[X], \mathcal{R}_\rho(\mu) := \int x\mu(dx)$$



Risk measures

The axiomatic approach to risk measures $(+, \geq)$ was introduced by Artzner, Delbaen, Eber and Heath '99 . One of the main axioms is that ρ is convex.

$$\mathbb{V}[X] := \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$ES_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VaR_\alpha(X)d\alpha$$

$$HG_\alpha(X) := \inf_{m\in\mathbb{R}}\{m + \|(X-m)^+\|_{\Phi_\alpha}\}$$

$$GP^\lambda(X) := \mathbb{E}[X] + \lambda\mathbb{E}[|X^* - X^{**}|]$$

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The goal of our project is to explore abstract conditions $(+, \geq)$ under which convex, law-invariant functionals are statistical consistent.

Automatic continuity of positive linear functionals

Theorem (Krein, 1940)

Let $\mathcal{X}$ be a Banach lattice. Then any positive linear functional $f : \mathcal{X} \to \mathbb{R}$ is continuous.

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Definition

Let $\mathcal{X}$ be a Riesz space, (X_n) be a sequence in $\mathcal{X}$ and $X \in \mathcal{X}$. We say that (X_n) converges uniformly to $X \in \mathcal{X}$ and write $X_n \xrightarrow{u} X$ iff

$$\exists V \in \mathcal{X}_+ \ \ \forall \epsilon > 0 \ \ \exists n_0 \ \ \forall n \geq n_0 \ \ |X_n - X| \leq \epsilon V$$

We also say that V is the regulator of the uniform convergence.

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Definition

Let $\mathcal{X}$ be a Riesz space. We say that a functional $f : \mathcal{X} \to \mathbb{R}$ is almost order bounded above if for any $V \in \mathcal{X}_+$ and $X \in \mathcal{X}$ there exists $\lambda > 0$ and $w \in \mathbb{R}$ such that $f(X + \lambda Y) \leq w$ for all $Y \in [-V, V]$.

Linearity can be relaxed to convexity

Theorem (G-M-X)

Let $\mathcal{X}$ be a Riesz space and $f : \mathcal{X} \to \mathbb{R}$ be a convex functional that is almost order bounded above . Then for every sequence (X_n) in $\mathcal{X}$ and $X \in \mathcal{X}$ such that $X_n \xrightarrow{u} X$ we have that $f(X_n) \to f(X)$.

Sketch proof

Let V be the regulator of the uniform convergence $X_n \xrightarrow{u} X$ and fix $0 < \epsilon < 1$. Since f is almost order bounded above , there exists $w \in \mathbb{R}$ and $\lambda > 0$ such that $f(X + \lambda Y) \leq w$ for all $Y \in [-V, V]$. Thus we have

$$f(X + \lambda Y) \leq w \leq f(X) + |w - f(X)| \tag{1}$$

We fix also $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda\epsilon}|X_n - X| \in [-V, V] \tag{2}$$

for all $n \geq n_0$.

Note that

$$X_n = (1-\epsilon)X + \epsilon(X + \frac{1}{\epsilon}(X_n - X))$$

By the convexity of f we have

$$f(X_n) \leq (1-\epsilon)f(X) + \epsilon f(x + \frac{1}{\epsilon}(X_n - X)) \tag{3}$$

and

$$f(2X - X_n) \leq (1-\epsilon)f(X) + \epsilon f(X + \frac{1}{\epsilon}(X - X_n)) \tag{4}$$

Therefore by (3) we get

$$f(X_n) - f(X) \leq \epsilon\Big(f(X + \frac{1}{\epsilon}(X_n - X)) - f(X)\Big) \tag{5}$$

Thus by (1), (2) and (5) we get for all $n \geq n_0$ that

$$f(X_n) - f(X) \leq \epsilon\Big(f(X) + |w - f(X)| - f(X)\Big) \leq \epsilon|w - f(X)| \tag{6}$$

Topological Version

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A linear topology τ on a Riesz space is said locally solid if τ has a basis for zero consisting of solid sets. A Frechét space is a Riesz space equipped with a complete metrizable locally solid topology.

Lemma

Let $(\mathcal{X}, \tau)$ be a Frechét space and (X_n) in $\mathcal{X}$ such that $X_n \xrightarrow{\tau} X$, then there exists a subsequence (X_{k_n}) of (X_n) such that $X_{k_n} \xrightarrow{u} X$.

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Corollary

Let $(\mathcal{X}, \tau)$ be a Frechét space and $f : \mathcal{X} \to \mathbb{R}$ be a convex functional. Then the following are equivalent

(i) *f is almost order bounded above.*

(ii) *f is continuous.*

Proof.

$(ii) \Rightarrow (i)$: Let $V \in \mathcal{X}_+$ and $X \in \mathcal{X}$. Since f is continuous at X, we can find a neighbourhood $\mathcal{V}$ of 0 and w in $\mathbb{R}$ such that $f(X+Y) \leq w$ for all $Y \in \mathcal{V}$. The topological boundedness of $[-V, V]$ ensures that there exists $\lambda > 0$ such that $[-V, V] \subseteq \frac{1}{\lambda}\mathcal{V}$. Thus for any $Y \in [-V, V]$ we have that $f(X + \lambda Y) \leq w$. In particular, f is almost order bounded above. □

Orlicz space framework

A non constant function $\Phi : [0, \infty) \to [0, \infty]$ is said to be an *Orlicz function* if it is non decreasing, left-continuous, and satisfies $\Phi(0) = 0$. The Young class of the Orlicz function Φ is denoted by

$$Y^{\Phi} := \left\{X \in L^0\, ;\ \mathbb{E}\left[\Phi\left(|X|\right)\right] < \infty\right\}.$$

The Orlicz space L^{Φ} and the Orlicz heart H^{Φ} associated with Φ are defined as follows

$$L^{\Phi} := \left\{X \in L^0\, ;\ \mathbb{E}\left[\Phi\left(k|X|\right)\right] < \infty \text{ for some } k \in \mathbb{N}\right\}.$$

$$H^{\Phi} := \left\{X \in L^0\, ;\ \mathbb{E}\left[\Phi\left(k|X|\right)\right] < \infty \text{ for all } k \in \mathbb{N}\right\}.$$

$$H^{\Phi} \subset Y^{\Phi} \subset L^{\Phi}$$

We say that Φ satisfies the Δ_2-condition whenever there are $C, x_0 > 0$ such that $\Phi(2x) \leq C\Phi(x)$ for all $x \geq x_0$.

Statistical Robustness

Let Φ be an Orlicz function.
For a sequence $(X_n) \subset Y^\Phi$ and $X \in Y^\Phi$ we say that (X_n) Φ-converges in distribution to X and write $X_n \xrightarrow{\Phi\text{-}dist.} X$ whenever

$$X_n \xrightarrow{dist.} X \quad \text{and} \quad \mathbb{E}[\Phi(|X_n|)] \to \mathbb{E}[\Phi(|X|)].$$

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Definition (Krätschmer-Schied-Zähle 2014)

Let $\rho : Y^\Phi \to \mathbb{R}$ be a law-invariant functional. We say that ρ statistical robust at $X \in Y^\Phi$ whenever ρ is continuous at $X \in Y^\Phi$ with respect to the Φ-convergence in distribution, that is $X_n \xrightarrow{\Phi\text{-}dist.} X \Rightarrow \rho(X_n) \to \rho(X)$.

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Remark

The (LLN) implies that if ρ is statistical robust then is also statistical consistent.

Order convergence vs Φ-convergence in distribution

Let $\mathcal{X}$ be a subset of L^0, (X_n) be a sequence in $\mathcal{X}$ and $X \in \mathcal{X}$, we write

$$X_n \xrightarrow{o} X \text{ in } \mathcal{X} \quad :\Longleftrightarrow \quad X_n \xrightarrow{\text{a.s.}} X \quad \text{and} \quad \sup |X_n| \in \mathcal{X}.$$

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Lemma

Let $(X_n) \subset Y^{\Phi}$ and $X \in Y^{\Phi}$ such that $X_n \xrightarrow{\Phi\text{-dist.}} X$. Then there exists a subsequence (X_{k_n}) of (X_n), a sequence $(Y_n) \subset Y^{\Phi}$ and $Y \in Y^{\Phi}$ such that Y_n has the same law as X_{k_n}, Y has the same law as X and $Y_n \xrightarrow{o} Y$ in Y^{Φ}.

Sketch proof

Since our probability space is non-atomic, the classical Skorohod representation yields $(Y_n) \subset Y^\Phi$ and $Y \in Y^\Phi$ such that $Y \sim X$, and $Y_n \sim X_n$ for every $n \in \mathbb{N}$, and $Y_n \xrightarrow{a.s.} Y$. Clearly,

$$\lim \mathbb{E}[\Phi(|Y_n|)] = \lim \mathbb{E}[\Phi(|X_n|)] = \mathbb{E}[\Phi(|X|)] = \mathbb{E}[\Phi(|Y|)] < \infty. \quad (7)$$

Since Φ is an Orlicz function, we also have that $\Phi(|Y_n|) \xrightarrow{a.s.} \Phi(|Y|)$. This combined with (7) yields that $\Phi(|Y_n|) \xrightarrow{||\cdot||_1} \Phi(|Y|)$. Thus by passing to a subsequence we may assume that $\mathbb{E}[\sup_{n\in\mathbb{N}} \Phi(|Y_n|)] < +\infty$. Since Φ is an Orlicz function we get that

$$\mathbb{E}[\Phi(\sup_{n\in\mathbb{N}} |Y_n|)] = \mathbb{E}[\sup_{n\in\mathbb{N}} \Phi(|Y_n|)] < +\infty.$$

In particular we have $Y_n \xrightarrow{o} Y$ in Y^Φ

Definition

Let $\mathcal{X} \subset L^0$ and $\rho : \mathcal{X} \to \mathbb{R}$,

- We say that ρ has the $\mathcal{X}$-Lebesgue property at $X \in \mathcal{X}$, whenever $X_n \xrightarrow{o} X$ in $\mathcal{X}$ implies that $\rho(X_n) \to \rho(X)$ for all (X_n) in $\mathcal{X}$.
- We say that ρ has the $\mathcal{X}$-Fatou property at $X \in \mathcal{X}$, whenever $X_n \xrightarrow{o} X$ in $\mathcal{X}$ implies that $\rho(X) \leq \liminf \rho(X_n)$ for all (X_n) in $\mathcal{X}$

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Corollary

Let Φ be an Orlicz function, $X \in Y^\Phi$ and $\mathcal{X} \subset L^0$ such that $Y^\Phi \subset \mathcal{X}$. Let $\rho : \mathcal{X} \to \mathbb{R}$ be a law-invariant functional and consider the continuity properties .

(i) *ρ is statistical robust at X.*

(ii) *ρ has the Y^Φ-Lebesgue property at X.*

(iii) *ρ is lower semi-continuous at X with respect to the Φ-convergence in distribution i.e. for every sequence $(X_n) \subset Y^\Phi$ and every $X \in Y^\Phi$ we have $X_n \xrightarrow{\Phi\text{-dist.}} X \implies \rho(X) \leq \liminf \rho(X_n)$.*

(iv) *ρ has the Y^Φ-Fatou property at X.*

Then (i) $\iff$ (ii) and (iii) $\iff$ (iv).

Statistical Robustness $\Longleftrightarrow$ Almost order bounded above

The following improves the celebrated result of Krätschmer-Schied-Zähle 2014.

Theorem (G-M-X)

Let Φ be an Orlicz function that satisfies the Δ_2 condition and $\rho : Y^\Phi \to \mathbb{R}$ be a convex, law-invariant functional. Then the following are equivalent.

(i) ρ *is almost order bounded above,*

(ii) ρ *is statistical robust.*

Proof.

We recall here that since Φ satisfies the Δ_2 condition we have that $Y^\Phi = H^\Phi = L^\Phi$. Also L^Φ is a Frechét space and the underlying topology is order continuous. Thus ρ is τ continuous if and only if ρ has the Y^Φ Lebesgue property. Now by applying our previous results we get the equivalence of (i) and (ii). $\square$

When Φ fails the Δ_2

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Theorem (Chen-Gao-Leung-Li 2022)

Let Φ be an Orlicz function and $\rho : L^\Phi \to \mathbb{R}$ be a convex, decreasing, law-invariant functional. Then ρ has the L^Φ-Fatou property at any $X \in L^\Phi$ such that $X^- \in H^\Phi$.

Corollary

Let Φ be an Orlicz function and $\rho : L^\Phi \to \mathbb{R}$ be a convex, decreasing, law-invariant functional. Then ρ is lower semi-continuous at any $X \in H^\Phi$ with respect to the Φ-convergence in distribution.

Statistical consistency

Proposition (M-G-X)

Let Φ be an Orlicz function $X \in L^{\Phi}$, and $\rho : L^{\Phi} \to \mathbb{R}$ be a law-invariant functional. If ρ has the L^{Φ}-Lebesgue property at X, then ρ is statistical consistent at X.

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Let Φ be an Orlicz function $X \in L^{\Phi}$, and $\rho : L^{\Phi} \to \mathbb{R}$ be a law-invariant functional. If ρ has the L^{Φ}-Lebesgue property at X, then ρ is statistical consistent at X.

Sketch proof WLOG we may assume that $X \in Y^{\Phi}$. Let μ_0 be the law of X, (X_n) be a sequence of independent identical random variables in L^0 with common law μ_0 and $\hat{m}_n$ the corresponding empirical distribution of (X_n). By an application of (LLN) we can find a measurable set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ we have $\int \Phi(|x|)\widehat{m}_n(\omega)(dx) = \frac{1}{n}\sum_{i=1}^{n} \Phi(|X_i|) \to \mathbb{E}[\Phi(|X|)] = \int \Phi(|x|)\mu(dx)$ and $m_n(\omega) \xrightarrow{dist.} \mu_0$. Since our probability space is non-atomic, we may find $(X_n^{\omega}), X \in L^0$ with law $(\mu_n(\omega))$ and μ_0 respectively. We then clearly have that $X_n^{\omega} \xrightarrow{\Phi\text{-}dist.} X$ and the continuity of ρ with respect to the Φ-convergence in distribution yields that $\widehat{\rho}_n(\omega) \to \rho(X)$ and thus ρ is strongly consistent at X.

Haezendonck-Goovaerts principle

For the following we fix a finite-valued convex Orlicz function Φ that is normalized by $\Phi(1) = 1$, a confidence level $\alpha \in (0, 1)$ and we set $\Phi_\alpha := \frac{\Phi}{1-\alpha}$.

Definition

The *Haezendonck-Goovaerts premium principle* associated to Φ at level α is the map $\pi_\alpha : L^\Phi \to \mathbb{R}$ defined by

$$HG_\alpha(X) := \inf_{m \in \mathbb{R}} \{m + \|(X - m)^+\|_{\Phi_\alpha}\}$$

- HG_α is convex, increasing, and law invariant.

Theorem (Ahn-Shyamalkumar 2014)

The Haezendonck-Goovaerts premium principle HG_α is statistical consistent at any $X \in L^\Phi$.

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Proposition (G-M-X)

The Haezendonck-Goovaerts premium principle HG_α, *restricted to* Y^Φ, *is lower semicontinuous with respect to* Φ*-convergence in distribution, i.e. for every sequence* $(X_n) \subset Y^\Phi$ *and every* $X \in Y^\Phi$ *we have*

$$X_n \xrightarrow{\Phi\text{-dist.}} X \implies HG_\alpha(X) \leq \liminf HG_\alpha(X_n).$$

Thank you for your attention!