Metric geometry on symmetric cones

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Symmetric cones

A symmetric cone C is the interior of a solid closed cone $\overline{C}$ in a finite dim. inner-product space $(V, \langle \cdot , \cdot \rangle)$

such that

1. $\mathrm{Aut}(C)$ acts transitively on C.
2. $\overline{C}$ is self dual.

Koecher-Vinberg: Symmetric cones are precisely the interiors of the cones of squares in Euclidean Jordan algebras

Euclidean Jordan algebras

A Euclidean Jordan algebra is an inner-product space $(V, \langle \cdot , \cdot \rangle)$ with a commutative bilinear product $x \bullet y$, such that

$$x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y) \quad \text{and} \quad \langle x \bullet y, z \rangle = \langle y, x \bullet z \rangle$$

$\mathrm{Herm}(n, \mathbb{C})_+ = \{A \in \mathrm{Herm}(n, \mathbb{C}) : A$ positive definite$\}$ and

$$A \bullet B = \frac{1}{2}(AB + BA)$$

Basics 1

The interior of the cone $\overline{C} = \{x^2 : x \in V\}$ consists of the invertible squares and contains the unit u.

Every $x \in V$ has a spectral decomposition, $x = \sum_{i=1}^{r} \lambda_i p_i$, where the p_i's are orthogonal primitive idempotents with $u = p_1 + \cdots + p_r$.

$\sigma(x) = \{\lambda_1, \ldots, \lambda_r\}$ is the spectrum of x, $\mathrm{tr}(x) = \lambda_1 + \cdots + \lambda_r$, and $\det(x) = \lambda_1 \cdots \lambda_r$

Functional calculus, e.g., $e^x = e^{\lambda_1} p_1 + \cdots + e^{\lambda_r} p_r$

Basics 2

For $x \in V$ the quadratic representation $Q_x : V \to V$ is given by $Q_x(y) = 2x \bullet (x \bullet y) - x^2 \bullet y$.

If x is inverible, then $Q_x \in \mathrm{Aut}(\mathrm{C})$

Jordan, Von Neumann, Wigner: There are 5 simple Euclidean Jordan algebras,

1. $\mathrm{Sym}(n, \mathbb{R})$ for $n \geq 3$
2. $\mathrm{Herm}(n, \mathbb{C})$ for $n \geq 3$
3. $\mathrm{Herm}(n, \mathbb{H})$ for $n \geq 3$
4. $\mathbb{R} \times \mathbb{R}^{n-1}$ for $n \geq 1$ (Spin factor)
5. $\mathrm{Herm}(3, \mathbb{O})$ exceptional one

Riemannian structure

A symmetric cone C has a Riemannian distance

$$d_R(x, y) = \|\log Q_{x^{-1/2}} y\|_2 = (\sum_{i=1}^{r} \lambda_i(z)^2)^{1/2}, \text{ where } z = \log Q_{x^{-1/2}} y$$

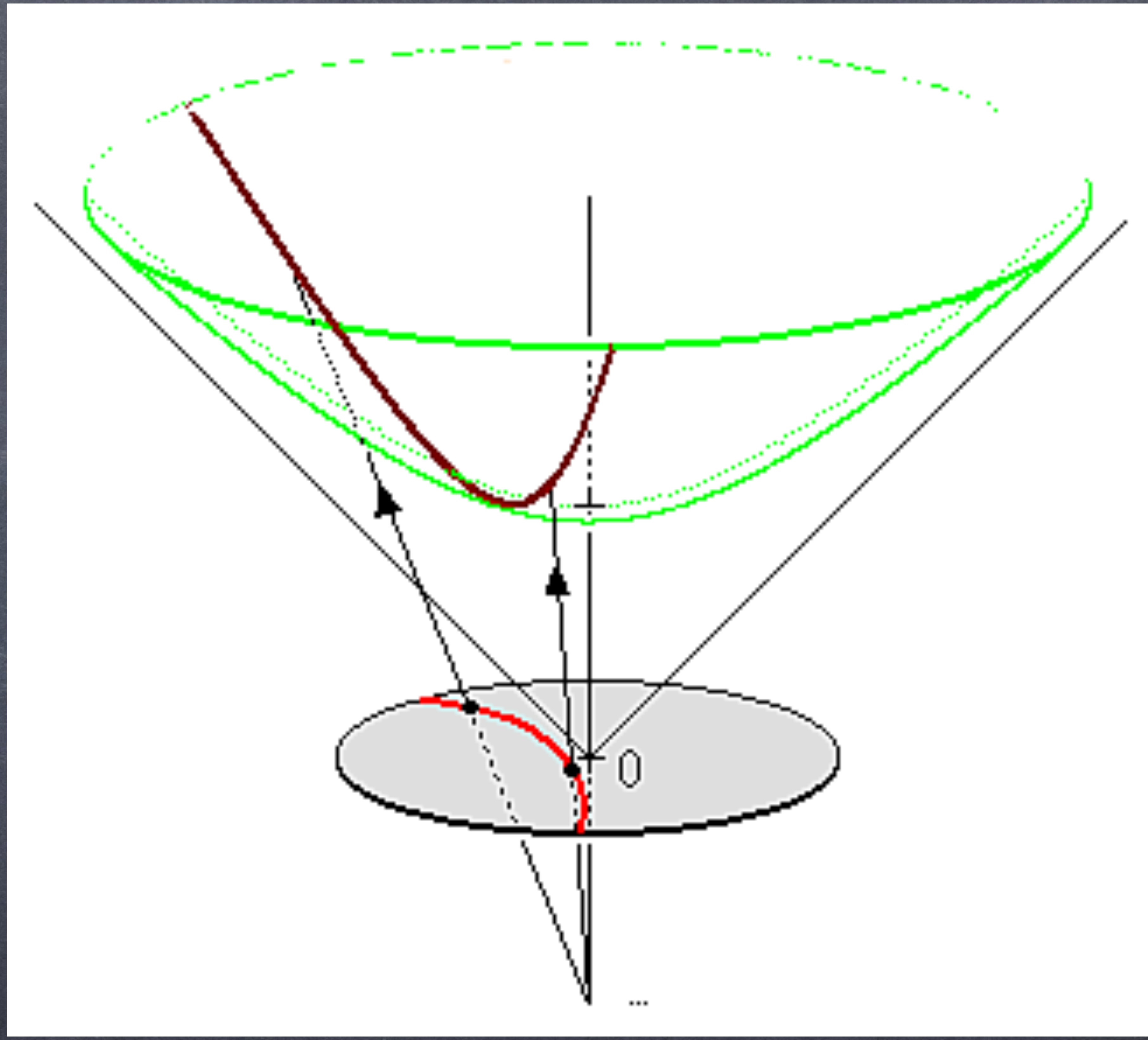
$$d_R(x, y) = \inf L(\gamma), \text{ where}$$

$$L(\gamma) = \int_0^1 \langle \gamma'(t), Q_{\gamma(t)^{-1}} \gamma'(t) \rangle^{1/2} \mathrm{d}t$$

inf over piecewise C^1-paths γ with $\gamma(0) = x$ and $\gamma(1) = y$

$\Omega_C = \{x \in C \colon \det(x) = 1\}$

Ω_C is a complete geodesic submanifold, hence also has a Riemannian structure



Hyperbolic plane $\Omega_{\mathrm{Sym}(2,\mathbb{R})_+}$

Noncompact type symmetric space

These are symmetric spaces, so for each $x \in C$ there exists an isometry $s_x \colon C \to C$ such that $s_x^2 = \mathrm{id}$ and x is an isolated fixed point of s_x

Symmetries, $s_x(y) = Q_x y^{-1}$

$\mathrm{Herm}(n, \mathbb{C})_+ = \{A \in \mathrm{Herm}(n, \mathbb{C}) : A$ positive definite$\}$ corresponds to $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$

$\Omega_{\mathrm{Herm}(n,\mathbb{C})}$ corresponds to $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$

Finsler distance

Finsler distance d_F on manifold M is length metric.

$d_F(x, y)$ is infimum of lengths, $L(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\,\mathrm{d}t$

over all piecewise C^1-paths $\gamma\colon [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$

Distance and cone order

For $x \in V$ and $y \in C$ let $M(x/y) = \inf\{\beta \in \mathbb{R} : x \leq \beta y\}$ and $m(x/y) = \sup\{\alpha \in \mathbb{R} : \alpha y \leq x\}$.

Symmetric cones: $M(x/y) = M(Q_{y^{-1/2}}x/u) = \max \sigma(Q_{y^{-1/2}}x)$

and

$$m(x/y) = m(Q_{y^{-1/2}}x/u) = \min \sigma(Q_{y^{-1/2}}x)$$

Thompson metric, $d_T(x, y) = \max\{\log M(x/y), \log M(y/x)\}$

On Ω_C, Hilbert metric, $d_H(x, y) = \log M(x/y) + \log M(y/x)$

Finsler structure

Thompson metric: For $x \in C$ the norm in $T_x C \cong V$,

$$\|z\|_x = \inf\{\lambda : -\lambda x \leq z \leq \lambda x\} = \max\{M(z/x), M(-z/x)\}$$

Hilbert metric: For $u \in \Omega_C$ the norm in

$T_u \Omega_C \cong \{z \in V : \mathrm{tr}(z) = 0\}$ is given by

$$|z|_u = M(z/u) - m(z/u) = \operatorname{diam} \sigma(z)$$

For general $x \in \Omega_C$ the tangent space is given by

$Q_{x^{1/2}}(T_u \Omega_C)$ and equipped with norm

$$|z|_x = |Q_{x^{-1/2}} z|_u$$

Compatifications

Compactifications of noncompact type symmetric spaces is a rich subject

1. Satake compactifications
2. Furstenberg compactifications
3. Martin compactifications

Satake compactifications and Martin compactifications can be realised as horofunction compactifications under suitable Finsler distances

Haettel, Schilling, Walsh, Wienhard

Horofunction compactification

(M, d) a metric space, fix basepoint $b \in M$

$C(M)$ space of continuous functions $f : M \to \mathbb{R}$
equipped with top. of pointwise convergence

$\mathrm{Lip}_b^1(M)$ set of 1-Lipschitz functions f in $C(M)$
with $f(b) = 0$, is a compact subset

For $y \in M$ let $h_y(x) = d(x, y) - d(b, y)$

$\iota : y \in M \to h_y \in \mathrm{Lip}_b^1(M)$

Horofunction compactification

$\overline{M}^h = \overline{\iota(M)}$ is a compact set, which is called the horofunction compactification of M

$\partial\overline{M}^h = \overline{\iota(M)} \backslash \iota(M)$ is called the horoboundary and its elements are called horofunctions

If (M, d) is a proper geodesic metric space, then h is a horofunction iff there exists a sequence (y^n) in M with $d(y^n, b) \to \infty$ and $h_{y^n} \to h$

Horofunctions

For (C, d_T) we have horofunctions,

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\},$$

where $y, z \in \overline{C}$ with $\max\{\|y\|_u, \|z\|_u\} = 1$ and $y \bullet z = 0$.
(If $y = 0$ or $z = 0$, we omit the corresponding term)

For (Ω_C, d_H) we have horofunctions,

$$h(x) = \log M(y/x) + \log M(z/x^{-1})$$

where $y, z \in \partial\overline{C}$ with $\|y\|_u = 1, \|z\|_u = 1$ and $y \bullet z = 0$

L. Lins, Nussbaum, Wortel J. Anal. Math 2018

Geometry of horoboundary

For horofunctions h and h' we say that $h \sim h'$ if

$$\sup_{x \in M} |h(x) - h'(x)| < \infty .$$

This gives a partition of the horoboundary into equivalence classes.

Can we describe the global topology and geometry of the horofunction compactification in concrete way?

Yes, we can view it as the dual unit ball of the Finsler norm in the tangent space at u

Equivalence classes for Thompson metric

$v, w \in \overline{C}$ are comparable if there exist $0 < \alpha \leq \beta$ such that $\alpha v \leq w \leq \beta v$.

$$h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\}$$

and

$$h'(x) = \max\{\log M(y'/x), \log M(z'/x^{-1})\}$$

are equivalent iff y is comparable to y' and z is comparable to z'.

Equivalence classes for Hilbert metric

$$h(x) = \log M(y/x) + \log M(z/x^{-1})$$

and

$$h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$$

are equivalent iff y is comparable to y' and z is comparable to z'.

Dual ball for C

The dual space of an order-unit space V is a base norm space, with unit ball $B_T^* = \mathrm{conv}(S(V) \cup -S(V))$.

Here $S(V) = \{\psi \in V_+^* : \psi(u) = 1\}$ is state space.

Using $x \in V \mapsto \langle \cdot , x \rangle \in V^*$ we identify V^* with V.

Closed boundary faces of $B_T^* \subset V$ are sets of the form

$$F_{p,q} = \mathrm{conv}((Q_p(V) \cap S(V)) \cup (Q_q(V) \cap -S(V))$$

where p and q are distinct orthogonal idempotents.

Edwards and Ruttimann

Homeomorphism $\varphi_T : \overline{C}^h \to B_T^*$

$$\varphi_T(x) = \frac{x - x^{-1}}{\mathrm{tr}(x + x^{-1})} \quad \text{for } x \in C$$

$$\varphi_T(h) = \frac{y - z}{\mathrm{tr}(y + z)} \text{ for } h(x) = \max\{\log M(y/x), \log M(z/x^{-1})\},$$

is a homeomorphism which maps $[h]$ onto $\mathrm{relint}\, F_{p,q}$.

Dual ball for Ω_C

What does dual ball B_H^* look like for Ω_C?

Tangent space $T_u = \{z \in V\colon \operatorname{tr}(z) = 0\}$ with norm

$$|z|_u = \operatorname{diam} \sigma(z)$$

$$B_H^* = 2\,\operatorname{conv}(S(V) \cup -S(V)) \cap T_u$$

Homeomorphism $\varphi_H \colon \overline{\Omega_C}^h \to B_H^*$

$$\varphi_H(x) = \frac{x}{\operatorname{tr}(x)} - \frac{x^{-1}}{\operatorname{tr}(x^{-1})} \quad \text{for } x \in \Omega_C$$

$$\varphi_H(h) = \frac{y}{\operatorname{tr}(y)} - \frac{z}{\operatorname{tr}(z)} \quad \text{for } h(x) = \log M(y/x) + \log M(z/x^{-1})$$

is a homeomorphism which maps $[h]$ onto $\operatorname{relint} 2F_{p,q} \cap T_u$.

Open problems

For which noncompact type symmetric spaces X with invariant Finsler metric d_F is the horofunction compactification naturally homeomorphic to the dual unit ball of the Finsler norm in the tangent space?

For which finite dimensional normed spaces is the horofunction compactification naturally homeomorphic to the closed dual unit ball?

(Kapovich & Leeb, Geom&Top. 2018)

Thanks

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