Positivity and well-posedness of a class of linear systems

ABDELAZIZ RHANDI
(University of Salerno)

POSI+IVITY XI, Ljubljana, Slovenia, July 10-14, 2023

Joint work with Y. El gantouh

Infinite-dimensional control systems

Infinite-dimensional control systems

We are interested in positivity and well-posedness of infinite-dimensional control systems described as

Infinite-dimensional control systems

We are interested in positivity and well-posedness of infinite-dimensional control systems described as

$$\begin{cases} \dot{z}(t) = A_m z(t), & t \geq 0, \\ z(0) = x, & \\ (G - \Gamma) z(t) = K u(t), & t \geq 0, \end{cases} \tag{1}$$

where $A_m : D(A_m) \subset X \to X$ is closed with $\overline{D(A_m)} = X$, $K \in \mathcal{L}(U, \partial X)$ (X, U, ∂X are Banach lattices), and $G, \Gamma : (D(A_m), \| \cdot \|_{A_m}) \to \partial X$ are linear continuous.

Infinite-dimensional control systems

We are interested in positivity and well-posedness of infinite-dimensional control systems described as

$$\begin{cases} \dot{z}(t) = A_m z(t), & t \geq 0, \\ z(0) = x, & \\ (G - \Gamma) z(t) = K u(t), & t \geq 0, \end{cases} \tag{1}$$

where $A_m : D(A_m) \subset X \to X$ is closed with $\overline{D(A_m)} = X$, $K \in \mathcal{L}(U, \partial X)$ (X, U, ∂X are Banach lattices), and $G, \Gamma : (D(A_m), \|\cdot\|_{A_m}) \to \partial X$ are linear continuous.
To (1) we associate the operator

$$\mathcal{A} := A_m, \qquad D(\mathcal{A}) = \{x \in D(A_m) : \quad (G - \Gamma)x = 0\}.$$

Assumptions

Assumptions

(H1) $A = (A_m)_{| \ker G}$ is a densely defined resolvent positive operator on X s.t. $\exists \mu_0 > s(A)$, $\forall \mu > \mu_0$, $\exists c(\mu) > 0$ with

$$\|R(\mu, A)x\| \geq c(\mu)\|x\|, \qquad (x \in X_+), \tag{2}$$

Assumptions

(H1) $A = (A_m)_{| \ker G}$ is a densely defined resolvent positive operator on X s.t. $\exists \mu_0 > s(A)$, $\forall \mu > \mu_0$, $\exists c(\mu) > 0$ with

$$\|R(\mu, A)x\| \geq c(\mu)\|x\|, \qquad (x \in X_+), \tag{2}$$

(H2) G is surjective.

Known results

Known results

Here instead of **(H1)** one assumes that

$A = A_m$ with domain $D(A) := \ker G$ generates a C_0-semigroup on X, then

- $\Gamma = 0$: wellposedness of (1) is obtained by the theory of well-posed linear systems, see O.J. Staffans 2005.

Known results

Here instead of **(H1)** one assumes that

$A = A_m$ with domain $D(A) := \ker G$ generates a C_0-semigroup on X, then

- $\Gamma = 0$: wellposedness of (1) is obtained by the theory of well-posed linear systems, see O.J. Staffans 2005.
- $K = 0$, $\Gamma \in \mathcal{L}(X, \partial X)$: wellposedness of (1) is obtained by G. Greiner 1987.

Known results

Here instead of **(H1)** one assumes that

$A = A_m$ with domain $D(A) := \ker G$ generates a C_0-semigroup on X, then

- $\Gamma = 0$: wellposedness of (1) is obtained by the theory of well-posed linear systems, see O.J. Staffans 2005.
- $K = 0$, $\Gamma \in \mathcal{L}(X, \partial X)$: wellposedness of (1) is obtained by G. Greiner 1987.
- $K = 0$, and admissibility conditions imply wellposedness of (1), see S. Hadd, R. Manzo and A. Rh. 2015.

The Greiner approach

The Greiner approach

- **(H1)** $\Longrightarrow$ A generates a positive C_0-semigroup on X, see W. Arendt 1987.

The Greiner approach

- **(H1)** $\Longrightarrow$ A generates a positive C_0-semigroup on X, see W. Arendt 1987.
- **(H2)** $\Longrightarrow$ $D(A_m) = D(A) \oplus \ker(\mu I_X - A_m)$, $\mu > s(A)$, and *the Dirichlet operator*

The Greiner approach

- **(H1)** $\Longrightarrow$ A generates a positive C_0-semigroup on X, see W. Arendt 1987.
- **(H2)** $\Longrightarrow$ $D(A_m) = D(A) \oplus \ker(\mu I_X - A_m),\ \mu > s(A)$, and *the Dirichlet operator*

$$D_\mu := \left(G_{|_{\ker(\mu I_X - A_m)}} \right)^{-1} \in \mathcal{L}(\partial X, \ker(\mu I_X - A_m)),$$

see G. Greiner 1987.

Resolvent positive operators and extrapolation

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$. We say that $y \in X_{-1}$ is positive, if $y \in \overline{X_+}^{\|\cdot\|_{-1}}$.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$. We say that $y \in X_{-1}$ is positive, if $y \in \overline{X_+}^{\|\cdot\|_{-1}}$. Then, $X_+ = X \cap (X_{-1})_+$, where $(X_{-1})_+ := \{y \in X_{-1} : y \geq 0\}$.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$. We say that $y \in X_{-1}$ is positive, if $y \in \overline{X_+}^{\|\cdot\|_{-1}}$. Then, $X_+ = X \cap (X_{-1})_+$, where $(X_{-1})_+ := \{y \in X_{-1} : y \geq 0\}$.
- $(\lambda - A_{-1})^{-1}(X_{-1})_+ \subseteq (X_{-1})_+,\ \forall \lambda > s(A_{-1}) = s(A)$ and

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$. We say that $y \in X_{-1}$ is positive, if $y \in \overline{X_+}^{\|\cdot\|_{-1}}$. Then, $X_+ = X \cap (X_{-1})_+$, where $(X_{-1})_+ := \{y \in X_{-1} : y \geq 0\}$.
- $(\lambda - A_{-1})^{-1}(X_{-1})_+ \subseteq (X_{-1})_+$, $\forall \lambda > s(A_{-1}) = s(A)$ and $B \in \mathcal{L}(X, X_{-1})$ is positive iff $\left[(\lambda - A_{-1})^{-1}B\right] X_+ \subseteq X_+$ for any large λ.

Resolvent positive operators and extrapolation

- **(H1)** implies A generates a positive C_0-semigroup on X, and $s(A) = \omega_0(A)$, cf. W. Arendt 1987.
- Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. The extrapolation space X_{-1} is the completion of X with respect to $\|x\|_{-1} := \|(\lambda - A)^{-1}\|$, $x \in X$. We say that $y \in X_{-1}$ is positive, if $y \in \overline{X_+}^{\|\cdot\|_{-1}}$. Then, $X_+ = X \cap (X_{-1})_+$, where $(X_{-1})_+ := \{y \in X_{-1} : y \geq 0\}$.
- $(\lambda - A_{-1})^{-1}(X_{-1})_+ \subseteq (X_{-1})_+$, $\forall \lambda > s(A_{-1}) = s(A)$ and $B \in \mathcal{L}(X, X_{-1})$ is positive iff $\left[(\lambda - A_{-1})^{-1}B\right] X_+ \subseteq X_+$ for any large λ.

See A. Batkai, B. Jacob, J. Voigt and J. Wintermayr 2018.

Infinite-dimensional control systems

Infinite-dimensional control systems

Define *the "control operator"*

$$B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}),$$

Infinite-dimensional control systems

Define *the "control operator"*

$$B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}),$$

and *the "observation operator"*

$$C = \Gamma|_{D(A)}.$$

Infinite-dimensional control systems

Define *the "control operator"*

$$B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}),$$

and *the "observation operator"*

$$C = \Gamma|_{D(A)}.$$

Problem (1) (with $K = 0$) can be written as a boundary input-output linear system

Infinite-dimensional control systems

Define *the "control operator"*

$$B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}),$$

and *the "observation operator"*

$$C = \Gamma|_{D(A)}.$$

Problem (1) (with $K = 0$) can be written as a boundary input-output linear system

$$\begin{cases} \dot{z}(t) = A_{-1}z(t) + Bv(t), & t \geq 0,\ z(0) = x, \\ Gz(t) = v(t), & t \geq 0, \\ y(t) = \Gamma z(t), & t \geq 0, \end{cases}$$

with the feedback law " $v = y$ ".

The Salamon-Weiss approach

The Salamon-Weiss approach

- Define $\Phi_t^A v := \int_0^t T_{-1}(t-s)Bv(s)\,ds,\ v \in L^1(\mathbb{R}_+;\partial X)$. B is called L^1-admissible if

$$\Phi_t^A \in \mathcal{L}(L^1(\mathbb{R}_+;\partial X), X),\ \forall t \geq 0.$$

The Salamon-Weiss approach

- Define $\Phi_t^A v := \int_0^t T_{-1}(t-s)Bv(s)\,ds$, $v \in L^1(\mathbb{R}_+;\partial X)$. B is called L^1-admissible if

$$\Phi_t^A \in \mathcal{L}(L^1(\mathbb{R}_+;\partial X), X),\ \forall t \geq 0.$$

- C is called L^1-admissible if $\exists \alpha > 0$ s.t.

$$\int_0^\alpha \|CT(t)x\|\,dt \leq \gamma\|x\|,\ \forall x \in X,$$

and some $\gamma = \gamma(\alpha) > 0$.

The Salamon-Weiss approach

The Salamon-Weiss approach

Remark: The mild solution of the above differential equation is

$$z(t) = T(t)x + \Phi_t^A v.$$

The Salamon-Weiss approach

Remark: The mild solution of the above differential equation is

$$z(t) = T(t)x + \Phi_t^A v.$$

Hence, the output function

$$y(t) = \Gamma z(t) = CT(t)x + \Gamma\Phi_t^A v =: (\Psi x)(t) + (\mathbb{F}v)(t)$$

The Salamon-Weiss approach

Remark: The mild solution of the above differential equation is

$$z(t) = T(t)x + \Phi_t^A v.$$

Hence, the output function

$$y(t) = \Gamma z(t) = CT(t)x + \Gamma\Phi_t^A v =: (\Psi x)(t) + (\mathbb{F}v)(t)$$

is well-defined for $x \in D(A),\ v \in W_0^{1,1}([0,t],\partial X)$.

The Salamon-Weiss approach

The Salamon-Weiss approach

Well-posed linear system: (A, B, C) is called well-posed if B, C are L^1-admissible operators and $\mathbb{F} \in \mathcal{L}(L^1([0,t], \partial X))$ for any $t > 0$.

The Salamon-Weiss approach

Well-posed linear system: (A, B, C) is called well-posed if $B,\ C$ are L^1-admissible operators and $\mathbb{F} \in \mathcal{L}(L^1([0,t],\partial X))$ for any $t > 0$.

Regular linear system: A Well-posed linear system (A, B, C) is called regular if

$$\mathrm{Range}(D_\mu) \subseteq D(C_\Lambda)$$

for some (and hence for all) $\mu \in \rho(A)$, where

The Salamon-Weiss approach

Well-posed linear system: (A, B, C) is called well-posed if $B,\ C$ are L^1-admissible operators and $\mathbb{F} \in \mathcal{L}(L^1([0,t], \partial X))$ for any $t > 0$.

Regular linear system: A Well-posed linear system (A, B, C) is called regular if

$$\text{Range}(D_\mu) \subseteq D(C_\Lambda)$$

for some (and hence for all) $\mu \in \rho(A)$, where

$$\begin{aligned} D(C_\Lambda) &:= \{x \in X : \lim_{\mu \to +\infty} C\mu R(\mu, A)x \text{ exists in } \partial X\}, \\ C_\Lambda x &:= \lim_{\mu \to +\infty} C\mu R(\mu, A)x, \qquad x \in D(C_\Lambda). \end{aligned}$$

The Salamon-Weiss approach

Well-posed linear system: (A, B, C) is called well-posed if $B,\ C$ are L^1-admissible operators and $\mathbb{F} \in \mathcal{L}(L^1([0,t], \partial X))$ for any $t > 0$.

Regular linear system: A Well-posed linear system (A, B, C) is called regular if

$$\text{Range}(D_\mu) \subseteq D(C_\Lambda)$$

for some (and hence for all) $\mu \in \rho(A)$, where

$$\begin{aligned} D(C_\Lambda) &:= \{x \in X : \lim_{\mu \to +\infty} C\mu R(\mu, A)x \text{ exists in } \partial X\}, \\ C_\Lambda x &:= \lim_{\mu \to +\infty} C\mu R(\mu, A)x, \qquad x \in D(C_\Lambda). \end{aligned}$$

see G. Weiss 1994.

Regular linear systems

Regular linear systems

Assume that (A, B, C) is a regular system. Then

Regular linear systems

Assume that (A, B, C) is a regular system. Then

- $D(A_m) \subseteq D(C_\Lambda)$ and $C_\Lambda x = \Gamma x$, $\forall x \in D(A_m)$.

Regular linear systems

Assume that (A, B, C) is a regular system. Then

- $D(A_m) \subseteq D(C_\Lambda)$ and $C_\Lambda x = \Gamma x$, $\forall x \in D(A_m)$.
- The operator $A_{-1} + BC_\Lambda$ with domain $D(A_{-1} + BC_\Lambda) := \{x \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)x \in X\}$ coincides with $\mathcal{A}$.

Regular linear systems

Assume that (A, B, C) is a regular system. Then

- $D(A_m) \subseteq D(C_\Lambda)$ and $C_\Lambda x = \Gamma x$, $\forall x \in D(A_m)$.
- The operator $A_{-1} + BC_\Lambda$ with domain $D(A_{-1} + BC_\Lambda) := \{x \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)x \in X\}$ coincides with $\mathcal{A}$. Recall

$$\mathcal{A} := A_m, \qquad D(\mathcal{A}) = \{x \in D(A_m) : \; Gx = \Gamma x\}\,.$$

Regular linear systems

Assume that (A, B, C) is a regular system. Then

- $D(A_m) \subseteq D(C_\Lambda)$ and $C_\Lambda x = \Gamma x$, $\forall x \in D(A_m)$.
- The operator $A_{-1} + BC_\Lambda$ with domain $D(A_{-1} + BC_\Lambda) := \{x \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)x \in X\}$ coincides with $\mathcal{A}$. Recall

$$\mathcal{A} := A_m, \qquad D(\mathcal{A}) = \{x \in D(A_m) : \; Gx = \Gamma x\}\,.$$

See S. Hadd, R. Manzo and A. Rh. 2015.

Main theorem

Main theorem

Assume

(A1) $\Gamma \geq 0$, $D_\mu \geq 0$, $\forall \mu$ large, **(H1)**, **(H2)**,

(A2) $C = \Gamma|_{D(A)}$ is L^1-admissible,

(A3) $\lim_{\lambda \to +\infty} \|\Gamma D_\lambda\| = 0$,

(A4) $\exists \mu_1 > s(A)$ s.t. $\sup_{\mu > \mu_1} \|\mu D_\mu v\| < \infty$, $\forall v \in \partial X$.

Main theorem

Assume

(A1) $\Gamma \geq 0$, $D_\mu \geq 0$, $\forall \mu$ large, **(H1)**, **(H2)**,

(A2) $C = \Gamma|_{D(A)}$ is L^1-admissible,

(A3) $\lim_{\lambda \to +\infty} \|\Gamma D_\lambda\| = 0$,

(A4) $\exists \mu_1 > s(A)$ s.t. $\sup_{\mu > \mu_1} \|\mu D_\mu v\| < \infty$, $\forall v \in \partial X$.

Then $\mathcal{A}$ generates a positive C_0-semigroup on X and $s(\mathcal{A}) = \omega(\mathcal{A})$.

Proof

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.
- (A4) $\Longrightarrow$ $\mathbb{F} \in \mathcal{L}(L^1([0,t], \partial X))$, $\forall t > 0$, where $\mathbb{F}v = \Gamma \Phi_t^A v$.

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.
- (A4) $\Longrightarrow$ $\mathbb{F} \in \mathcal{L}(L^1([0,t],\partial X)), \ \forall t > 0$, where $\mathbb{F}v = \Gamma\Phi_t^A v$.
- Since $A_{-1}x = A_m x - BGx, \ x \in D(A_m)$, one has $\lambda R(\lambda, A)D_0 = D_0 - D_\lambda$ (WLOG $s(A) < 0$) and hence,

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.
- (A4) $\Longrightarrow$ $\mathbb{F} \in \mathcal{L}(L^1([0,t],\partial X))$, $\forall t > 0$, where $\mathbb{F}v = \Gamma\Phi_t^A v$.
- Since $A_{-1}x = A_m x - BGx$, $x \in D(A_m)$, one has $\lambda R(\lambda, A)D_0 = D_0 - D_\lambda$ (WLOG $s(A) < 0$) and hence, $C\lambda R(\lambda, A)D_0 = \Gamma D_0 - \Gamma D_\lambda$, $\forall \lambda > s(A)$.

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.
- (A4) $\Longrightarrow$ $\mathbb{F} \in \mathcal{L}(L^1([0,t],\partial X))$, $\forall t > 0$, where $\mathbb{F}v = \Gamma\Phi_t^A v$.
- Since $A_{-1}x = A_m x - BGx$, $x \in D(A_m)$, one has $\lambda R(\lambda, A)D_0 = D_0 - D_\lambda$ (WLOG $s(A) < 0$) and hence, $C\lambda R(\lambda, A)D_0 = \Gamma D_0 - \Gamma D_\lambda$, $\forall \lambda > s(A)$. Thus, (A3) $\Longrightarrow$ Range$(D_0) \subseteq D(C_\Lambda)$.

Proof

- **(H1)** $\Longrightarrow$ B is L^1-admissible.
- (A4) $\Longrightarrow$ $\mathbb{F} \in \mathcal{L}(L^1([0,t],\partial X))$, $\forall t > 0$, where $\mathbb{F}v = \Gamma \Phi_t^A v$.
- Since $A_{-1}x = A_m x - BGx$, $x \in D(A_m)$, one has $\lambda R(\lambda, A)D_0 = D_0 - D_\lambda$ (WLOG $s(A) < 0$) and hence, $C\lambda R(\lambda, A)D_0 = \Gamma D_0 - \Gamma D_\lambda$, $\forall \lambda > s(A)$. Thus, (A3) $\Longrightarrow$ Range$(D_0) \subseteq D(C_\Lambda)$.
- (A, B, C) is regular.

Proof

Proof

- (A, B, C) is regular $\Longrightarrow \forall \mu > s(A),\ x \in D(\mathcal{A}),$

Proof

- (A, B, C) is regular $\Longrightarrow \forall \mu > s(A),\ x \in D(\mathcal{A})$,

$$\begin{aligned}(\mu I_X - \mathcal{A})x &= (\mu I_X - A_{-1} - B\Gamma)x \\ &= (\mu I_X - A_{-1})(I_X - D_\mu \Gamma)x \\ &= (\mu I_X - A)(I_X - D_\mu \Gamma)x,\end{aligned}$$

since $x - D_\mu \Gamma x \in D(A)$.

Proof

- (A, B, C) is regular $\Longrightarrow \forall \mu > s(A),\ x \in D(\mathcal{A})$,

$$\begin{aligned}(\mu I_X - \mathcal{A})x &= (\mu I_X - A_{-1} - B\Gamma)x \\ &= (\mu I_X - A_{-1})(I_X - D_\mu \Gamma)x \\ &= (\mu I_X - A)(I_X - D_\mu \Gamma)x,\end{aligned}$$

  since $x - D_\mu \Gamma x \in D(A)$.
- (A3) $\Longrightarrow \exists \mu_2 > s(A)$ s.t. $\|\Gamma D_{\mu_2}\| < 1$.

Proof

- (A, B, C) is regular $\Longrightarrow \forall \mu > s(A),\ x \in D(\mathcal{A})$,

$$\begin{aligned}(\mu I_X - \mathcal{A})x &= (\mu I_X - A_{-1} - B\Gamma)x \\ &= (\mu I_X - A_{-1})(I_X - D_\mu \Gamma)x \\ &= (\mu I_X - A)(I_X - D_\mu \Gamma)x,\end{aligned}$$

  since $x - D_\mu \Gamma x \in D(A)$.
- (A3) $\Longrightarrow \exists \mu_2 > s(A)$ s.t. $\|\Gamma D_{\mu_2}\| < 1$. Hence, $\|\Gamma D_\mu\| \leq \|\Gamma D_{\mu_2}\| < 1,\ \forall \mu \geq \mu_2$.

Proof

- (A, B, C) is regular $\Longrightarrow$ $\forall \mu > s(A),\ x \in D(\mathcal{A})$,

$$\begin{aligned}(\mu I_X - \mathcal{A})x &= (\mu I_X - A_{-1} - B\Gamma)x \\ &= (\mu I_X - A_{-1})(I_X - D_\mu \Gamma)x \\ &= (\mu I_X - A)(I_X - D_\mu \Gamma)x,\end{aligned}$$

  since $x - D_\mu \Gamma x \in D(A)$.
- (A3) $\Longrightarrow$ $\exists \mu_2 > s(A)$ s.t. $\|\Gamma D_{\mu_2}\| < 1$. Hence, $\|\Gamma D_\mu\| \leq \|\Gamma D_{\mu_2}\| < 1,\ \forall \mu \geq \mu_2$. Thus,

$$\begin{aligned}R(\mu, \mathcal{A}) &= (I_X - D_\mu \Gamma)^{-1} R(\mu, A) \\ &= R(\mu, A) + D_\mu \sum_{n=0}^{\infty} (\Gamma D_\mu)^n \Gamma R(\mu, A) \\ &\geq R(\mu, A),\ \forall \mu \geq \mu_2.\end{aligned}$$

Proof

Proof

- **(H1)** and $R(\mu, \mathcal{A}) \geq R(\mu, A) \implies$

Proof

- **(H1)** and $R(\mu, \mathcal{A}) \geq R(\mu, A) \implies$

$$\|R(\mu, \mathcal{A})x\| \geq \|R(\mu, A)x\| \geq c(\mu)\|x\|,$$

$\forall\, x \in X_+,\ \mu > \max\{\mu_0, \mu_2\}$.

Proof

- **(H1)** and $R(\mu, \mathcal{A}) \geq R(\mu, A) \implies$

$$\|R(\mu, \mathcal{A})x\| \geq \|R(\mu, A)x\| \geq c(\mu)\|x\|,$$

$\forall\, x \in X_+,\ \mu > \max\{\mu_0, \mu_2\}$.

- W. Arendt 1987 $\Longrightarrow$ the statement.

Boltzmann equation on a finite connected graph

Boltzmann equation on a finite connected graph

Consider the PDE (Σ_{TN})

$$\begin{cases} \frac{\partial}{\partial t} z_j(t,x,v) = v \frac{\partial}{\partial x} z_j(t,x,v) + q_j(x,v) z_j(t,x,v), & t \geq 0,\ (x,v) \in \Omega, \\ z_j(0,x,v) = f_j(x,v) \geq 0, & (x,v) \in \Omega, \\ \imath_{ij}^{out} z_j(t,1,\cdot) = \mathrm{w}_{ij} \sum_{k=1}^{M} \imath_{ik}^{inc} \mathbb{J}_k(z_k)(t,0,\cdot), & t \geq 0, \end{cases}$$

$i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, M\}$, $\Omega := [0,1] \times [v_{\min}, v_{\max}]$,

Boltzmann equation on a finite connected graph

Consider the PDE (Σ_{TN})

$$\begin{cases} \frac{\partial}{\partial t} z_j(t,x,v) = v\frac{\partial}{\partial x} z_j(t,x,v) + q_j(x,v) z_j(t,x,v), & t \geq 0,\ (x,v) \in \Omega, \\ z_j(0,x,v) = f_j(x,v) \geq 0, & (x,v) \in \Omega, \\ \imath_{ij}^{out} z_j(t,1,\cdot) = \mathrm{w}_{ij} \sum_{k=1}^{M} \imath_{ik}^{inc} \mathbb{J}_k(z_k)(t,0,\cdot), & t \geq 0, \end{cases}$$

$i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\}$, $\Omega := [0,1] \times [v_{\min}, v_{\max}]$,

$$\imath_{ij}^{out} := \begin{cases} 1, & \text{if } \mathsf{v}_i \bullet \xrightarrow{\mathsf{e}_j} \\ 0, & \text{if not,} \end{cases} \qquad \imath_{ij}^{inc} := \begin{cases} 1, & \text{if } \xrightarrow{\mathsf{e}_j} \bullet\, \mathsf{v}_i \\ 0, & \text{if not,} \end{cases}$$

Boltzmann equation on a finite connected graph

Consider the PDE (Σ_{TN})

$$\begin{cases} \frac{\partial}{\partial t} z_j(t,x,v) = v \frac{\partial}{\partial x} z_j(t,x,v) + q_j(x,v) z_j(t,x,v), & t \geq 0,\ (x,v) \in \Omega, \\ z_j(0,x,v) = f_j(x,v) \geq 0, & (x,v) \in \Omega, \\ \imath_{ij}^{out} z_j(t,1,\cdot) = \mathrm{w}_{ij} \sum_{k=1}^{M} \imath_{ik}^{inc} \mathbb{J}_k(z_k)(t,0,\cdot), & t \geq 0, \end{cases}$$

$i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\}$, $\Omega := [0,1] \times [v_{\min}, v_{\max}]$,

$$\imath_{ij}^{out} := \begin{cases} 1, & \text{if } \mathrm{v}_i \bullet\!\xrightarrow{\ \mathrm{e}_j\ } \\ 0, & \text{if not,} \end{cases} \qquad \imath_{ij}^{inc} := \begin{cases} 1, & \text{if } \xrightarrow{\ \mathrm{e}_j\ }\!\bullet\, \mathrm{v}_i \\ 0, & \text{if not,} \end{cases}$$

$0 < v_{\min} < v_{\max}$, $q_j \in L^\infty(\Omega)$.

(Σ_{TN}) as Cauchy problem

(Σ_{TN}) as Cauchy problem

$$X := \left(L^1(\Omega)\right)^M, \qquad \|\varphi\|_X := \sum_{j=1}^{M} \|\varphi_j\|_{L^1(\Omega)},$$

$$\partial X := \left(L^1(v_{\min}, v_{\max})\right)^N, \qquad \|g\|_{\partial X} := \sum_{i=1}^{N} \|f_i\|_{L^1(v_{\min}, v_{\max})},$$

$$\mathbb{W} := (W(\Omega))^M, \qquad \|f\|_{\mathbb{W}} = \|f\|_X + \|\partial_x f\|_X,$$

$$W(\Omega) := \{g \in L^1(\Omega) : \partial_x g \in L^1(\Omega)\}.$$

(Σ_{TN}) as Cauchy problem

(Σ_{TN}) as Cauchy problem

$$A_m f = v\partial_x f + q(\cdot,\cdot)f,\; D(A_m) = \left\{ f \in \mathbb{W} : f(1,v) \in \mathrm{Range}(\mathcal{J}_{\mathsf{w}}^{out})^{\top} \right\}.$$

(Σ_{TN}) as Cauchy problem

$$A_m f = v\partial_x f + q(\cdot,\cdot)f,\ D(A_m) = \left\{ f \in \mathbb{W} : f(1,v) \in \mathrm{Range}(\mathcal{J}_{\mathrm{w}}^{out})^{\top} \right\}.$$

$$\mathbb{J}_k(f_k)(x,v) = \int_{v_{\min}}^{v_{\max}} \ell_k(x,v,v')f_k(x,v')dv',\ (x,v) \in \Omega,\ \ f \in X.$$

(Σ_{TN}) as Cauchy problem

$$A_m f = v\partial_x f + q(\cdot,\cdot)f,\ D(A_m) = \left\{ f \in \mathbb{W} : f(1,v) \in \mathrm{Range}(\mathcal{I}^{out}_{\mathrm{w}})^{\top} \right\}.$$

$$\mathbb{J}_k(f_k)(x,v) = \int_{v_{\min}}^{v_{\max}} \ell_k(x,v,v') f_k(x,v')dv',\ (x,v) \in \Omega,\ \ f \in X.$$

$$Gf := \mathcal{I}^{out} f(1,v), \qquad \Gamma f := \mathcal{I}^{inc}(\mathbb{J}f)(0,v),\ v \in [v_{\min}, v_{\max}],\ f \in \mathbb{W},$$

(Σ_{TN}) as Cauchy problem

$$A_m f = v\partial_x f + q(\cdot,\cdot)f,\ D(A_m) = \left\{ f \in \mathbb{W} : f(1,v) \in \mathrm{Range}(\mathcal{I}_{\mathsf{w}}^{out})^\top \right\}.$$

$$\mathbb{J}_k(f_k)(x,v) = \int_{v_{\min}}^{v_{\max}} \ell_k(x,v,v')f_k(x,v')dv',\ (x,v) \in \Omega,\ \ f \in X.$$

$$Gf := \mathcal{I}^{out} f(1,v), \qquad \Gamma f := \mathcal{I}^{inc}(\mathbb{J}f)(0,v),\ v \in [v_{\min}, v_{\max}],\ \ f \in \mathbb{W},$$

$$\mathcal{I}_{\mathsf{w}}^{out} := (\mathsf{w}_{ij}\imath_{ij}^{out}),\ \mathcal{I}^{out} := (\imath_{ij}^{out}),\ \mathcal{I}^{inc} = (\imath_{ij}^{inc}),\ \mathbb{J} = \mathrm{diag}(\mathbb{J}_k).$$

(Σ_{TN}) as Cauchy problem

(Σ_{TN}) as Cauchy problem

One can rewrite (Σ_{TN}) as

$$\begin{cases} \dot{z}(t) = A_m z(t), & t \geq 0, \\ z(0) = f, \\ Gz(t) = \Gamma z(t), & t \geq 0. \end{cases}$$

Well-posedness

Well-posedness

One can see

Well-posedness

One can see

$$
\begin{aligned}
(D_\mu g)_j(x,v) &= e^{\int_x^1 \frac{q_j(\sigma,v)-\mu}{v}d\sigma} \sum_{i=1}^{N} w_{ij} g_i(v), \\
(R(\mu,A)f)_j(x,v) &= \int_x^1 e^{\int_x^y \frac{q_j(\sigma,v)-\mu}{v}d\sigma} \frac{1}{v} f_j(y,v)\,dy, \ \forall \mu > \max_j \|q_j\|_\infty .
\end{aligned}
$$

Well-posedness

One can see

$$(D_\mu g)_j(x,v) = e^{\int_x^1 \frac{q_j(\sigma,v)-\mu}{v}d\sigma} \sum_{i=1}^{N} w_{ij} g_i(v),$$

$$(R(\mu,A)f)_j(x,v) = \int_x^1 e^{\int_x^y \frac{q_j(\sigma,v)-\mu}{v}d\sigma} \frac{1}{v} f_j(y,v)dy, \ \forall \mu > \max_j \|q_j\|_\infty.$$

Main Theorem $\Longrightarrow$ well-posedness.

Many thanks for your attention