Functions operating on several multivariate distribution functions

Paul Ressel

Let \( I_1, \ldots, I_d \subseteq \mathbb{R} \) be non-degenerate intervals, \( I := I_1 \times \cdots \times I_d \), and let \( f : I \to \mathbb{R} \) be any function. For \( s \in I, h \in \mathbb{R}_+^d \) such that also \( s + h \in I \), put

\[
(E_h f)(s) := f(s + h)
\]

and \( \Delta_h := E_h - E_0 \), i.e. \( (\Delta_h f)(s) := f(s + h) - f(s) \). Since \( \{E_h \mid h \in \mathbb{R}_+^d\} \) is commutative (where defined), so is also \( \{\Delta_h \mid h \in \mathbb{R}_+^d\} \). In particular, with \( e_1, \ldots, e_d \) denoting standard unit vectors in \( \mathbb{R}^d \), \( \Delta_{h_1 e_1}, \ldots, \Delta_{h_d e_d} \) commute. As usual, \( \Delta_0 f := f \) (also for \( h = 0 \), but clearly \( \Delta_0 f = 0 \)). For \( \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d \) and \( h = (h_1, \ldots, h_d) \in \mathbb{R}_+^d \) we put

\[
\Delta^n_h := \Delta_{n_1 h_1 e_1} \Delta_{n_2 h_2 e_2} \cdots \Delta_{n_d h_d e_d},
\]

so that \( (\Delta^n_h f)(s) \) is defined for \( s, s + \sum_{i=1}^d n_i h_i e_i \in I \).

**Definition.** \( f : I \to \mathbb{R} \) is \( \mathbf{n} - \uparrow \) (read "\( \mathbf{n} \)-increasing") iff \( (\Delta^\mathbf{p}_h f)(s) \geq 0 \forall s \in I, h \in \mathbb{R}_+^d, \mathbf{p} \in \mathbb{N}_0^d, 0 \neq \mathbf{p} \leq \mathbf{n} \), such that \( s_j + p_j h_j \in I_j \forall j \in [d] \).
A specially important case is \( n = 1_d \); being \( 1_d \)-\( \uparrow \) is the "crucial" property of d.f.s. More precisely: \( f : I \to \mathbb{R}_+ \) is the d.f. of a (non-negative) measure \( \mu \), i.e. \( f(s) = \mu ([-\infty, s] \cap I) \) \( \forall s \in I \), if and only if \( f \) is \( 1_d \)-\( \uparrow \) and right-continuous; c.f. [9], Theorem 7.

Let us for a moment consider the case \( d = 1 \). Then \( I \subseteq \mathbb{R}, n = n \in \mathbb{N} \), we assume \( n \geq 2 \), and a famous old result of Boas and Widder ([1], Lemma 1) shows that a continuous function \( f : I \to \mathbb{R} \) is \( n \)-\( \uparrow \) (i.e. \( \Delta^j_h f \geq 0 \) \( \forall j \in [n], \forall h > 0 \)) iff

\[
(\Delta_h^{j_1} \Delta_h^{j_2} \ldots \Delta_h^{j_k} f)(s) \geq 0
\]

\( \forall j \in [n], \forall h_1, \ldots, h_j > 0 \) such that \( s, s+h_1+\ldots+h_j \in I \). For \( n = 2 \), \( f \) is \( 2 \)-\( \uparrow \) iff it is increasing and convex (and BTW automatically continuous on \( I \setminus \{ \text{sup } I \} \)).

The following definition now seems to be natural:

**Definition.** Let \( I_1, \ldots, I_d \subseteq \mathbb{R} \) be non-degenerate intervals, \( I = I_1 \times \cdots \times I_d \), \( f : I \to \mathbb{R} \), and \( k \in \mathbb{N} \). Then \( f \) is called \( k \)-increasing ("\( k \)-\( \uparrow \)"") iff \( \forall j \in [k], \forall h^{(1)}, \ldots, h^{(j)} \in \mathbb{R}_+^d, \forall s \in I \) such that \( s+h^{(1)}+\ldots+h^{(j)} \in I \)

\[
(\Delta_{h^{(1)}} \ldots \Delta_{h^{(j)}} f)(s) \geq 0.
\]

(We do not assume \( f \) to be continuous.)

We mentioned already that a univariate \( f \) is \( 2 \)-\( \uparrow \) iff it is increasing and convex. But also multivariate \( 2 \)-\( \uparrow \) functions are well-known: they are called ultramodular, mostly ultramodular aggregation functions, the latter meaning they are also increasing, and defined as functions \( f : [0,1]^d \to [0,1] \) with \( f(0_d) = 0 \) and \( f(1_d) = 1 \). Some simple properties of \( k \)-\( \uparrow \) functions are shown first.
Lemma 1. Let $f : [0, 1]^d \to \mathbb{R}$ be $2 \uparrow$. Then

(i) $f$ is continuous iff $f$ is continuous in $1_d$.

(ii) $f$ is right-continuous and on $[0, 1]^d$ continuous.

Our first theorem will state some equivalent conditions for $f$ to be $k \uparrow$. An essential ingredient will be positive linear (or affine) mappings: a linear function $\psi : \mathbb{R}^m \to \mathbb{R}^d$ is called positive iff $\psi(\mathbb{R}^m_+) \subseteq \mathbb{R}^d_+$; and an affine $\varphi : \mathbb{R}^m \to \mathbb{R}^d$ is positive iff its "linear part" $\varphi - \varphi(0)$ is.

Theorem 1. Let $I \subseteq \mathbb{R}^d$ be a non-degenerate interval, $f : I \to \mathbb{R}$, $k$, $d \in \mathbb{N}$. Then there are equivalent:

(i) $f$ is $k \uparrow$

(ii) $f$ is $n \uparrow \forall n \in \mathbb{N}^d_0$ with $0 < |n| \leq k$

(iii) $\forall m \in \mathbb{N}$, $\forall$ non-degenerate interval $J \subseteq \mathbb{R}^m$, $\forall$ positive affine $\varphi : \mathbb{R}^m \to \mathbb{R}^d$ such that $\varphi(J) \subseteq I$, also $f \circ \varphi$ is $k \uparrow$

(iv) $\forall m$, $J$, $\varphi$ as before, and $\forall n \in \mathbb{N}^m_0$ with $0 < |n| \leq k$ the function $f \circ \varphi$ is $n \uparrow$

(v) $\forall m$, $J$, $\varphi$ as before, and $\forall n \in \{0, 1\}^m$ with $0 < |n| \leq k$ the function $f \circ \varphi$ is $n \uparrow$.

Corollary 1. Let $I \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}$ be non-degenerate intervals. If $g : I \to B$ and $f : B \to \mathbb{R}$ are both $k \uparrow$, then so is $f \circ g$.

Theorem 2. Let $I \subseteq \mathbb{R}^{d_1}$ and $J \subseteq \mathbb{R}^{d_2}$ be non-degenerate intervals, $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, both non-negative and $k \uparrow$. Then also $f \otimes g$ is $k \uparrow$ on $I \times J$, and in case $I = J$ the product $f \cdot g$ is $k \uparrow$, too.
Proof. We first apply (ii) of Theorem 1. For \( 0 \neq (m, n) \in \mathbb{N}_0^{d_1} \times \mathbb{N}_0^{d_2}, \)
\((x, y) \in I \times J, h^{(1)} \in \mathbb{R}_+^{d_1}, h^{(2)} \in \mathbb{R}_+^{d_2}\) we have

\[
\left[ \Delta_{(h^{(1)}, h^{(2)})}^{m, n}(f \otimes g) \right](x, y) = \left( \Delta_{h^{(1)}}^{m} f \right)(x) \cdot \left( \Delta_{h^{(2)}}^{n} g \right)(y)
\]

and for \(|(m, n)| = |m| + |n| \leq k\) both factors on the RHS are non-negative. Since \(m = 0\) or \(n = 0\) is allowed, only \((m, n) \neq 0\) being required, we need in fact \(f \geq 0\) and \(g \geq 0\).

For \(I = J\) (with \(d_1 = d_2 =: d\)) let \(\varphi : \mathbb{R}^d \to \mathbb{R}^{2d}\) be given by \(\varphi(x) := (x, x)\), a positive linear map. Then \(\varphi(I) \subseteq I \times I\), and by Theorem 1 (iii) \((f \otimes g) \circ \varphi = f \cdot g\) is also \(k\)-\(\uparrow\).

We see that any monomial \(f(x) = \prod_{i=1}^d x_i^{n_i}\) (\(n_i \in \mathbb{N}\)) is \(k\)-\(\uparrow\) on \(\mathbb{R}_+^d\) for each \(k \in \mathbb{N}\). If \(c_i \in ]0, \infty[\) then \(\prod_{i=1}^d x_i^{c_i}\) is \(k\)-\(\uparrow\) on \(\mathbb{R}_+^d\) at least for \(c_i \geq k - 1, i = 1, \ldots, d\).

Examples 1. (a) For \(a > 0\) the function \(f(x, y) := (xy - a)_{+}\) is \(2\)-\(\uparrow\) on \(\mathbb{R}_+^2\), since \(t \mapsto (t - a)_{+}\) is \(2\)-\(\uparrow\) on \(\mathbb{R}_+\), by Corollary 1. In [11] on page 261 it was shown that \(f\) is not \((2, 1)\)-\(\uparrow\) (resp. \((1, 2)\)-\(\uparrow\)), but it is of course \((1, 1)\)-\(\uparrow\), and so a bivariate d.f.. The tensor product \(g(x, y) := (x - a)_{+} \cdot (y - b)_{+}\) is \((2, 2)\)-\(\uparrow\) \(\forall a, b > 0\), hence certainly \(2\)-\(\uparrow\), but not \(3\)-\(\uparrow\) since \(x \mapsto (x - a)_{+}\) is not.

Similarly \((xyz - a)^2_{+}\) is \(3\)-\(\uparrow\) on \(\mathbb{R}_+^3\), for \(a > 0\), and of course \((xy - a)^2_{+}\) is \(3\)-\(\uparrow\) on \(\mathbb{R}_+^2\). We’ll see later on that \(xy + xz + yz - xyz\) is \(2\)-\(\uparrow\) on \([0, 1]^3\), but not \(3\)-\(\uparrow\).

(b) Consider \(f_n(t) := t^n/(1+t)\) for \(t \geq 0\). It was shown in [6], Lemma 2.4, that \(f_n\) is \(n\)-\(\uparrow\) (it is not \((n+1)\)-\(\uparrow\)). So for any non-negative \(n\)-\(\uparrow\) function \(g\) on any interval in any dimension, \(g^n/(1 + g)\) is \(n\)-\(\uparrow\), too.
Approximation by Bernstein polynomials

The proof of our main result relies heavily on these special polynomials, since they inherit the monotonicity properties of interest. To define them we introduce for \( r \in \mathbb{N}, i \in \{0, 1, \ldots, r\} \)
\[
b_{i,r}(t) := \binom{r}{i} t^i (1-t)^{r-i}, \quad t \in \mathbb{R}
\]
and for \( i = (i_1, \ldots, i_d) \in \{0, 1, \ldots, r\}^d \)
\[
B_{i,r} := b_{i_1, r} \otimes \cdots \otimes b_{i_d, r}.
\]
For any \( f : [0,1]^d \to \mathbb{R} \) the associated Bernstein polynomials \( f^{(1)}, f^{(2)}, \ldots \) are defined by
\[
f^{(r)} := \sum_{0_d \leq i \leq r_d} f\left(\frac{i}{r}\right) \cdot B_{i,r}.
\]
It is perhaps not so well-known, that for each continuity point \( x \) of \( f \) we have
\[
f^{(r)}(x) \to f(x), \quad r \to \infty.
\]
In the following the "upper right boundary" of \([0, 1]^d\) will play a role. Let for \( \alpha \subseteq [d] \)
\[
T_\alpha := \{x \in [0,1]^d \mid x_i < 1 \iff i \in \alpha\}.
\]
Then \([0, 1]^d = \bigcup_{\alpha \subseteq [d]} T_\alpha, \ T_\emptyset = \{1_d\} \) and \( T_{[d]} = [0,1]^d \). The union \( \bigcup_{\alpha \subseteq [d]} T_\alpha \) is called the upper right boundary of \([0, 1]^d \).

**Theorem 3.** Let \( f : [0,1]^d \to \mathbb{R} \) have the property that each restriction \( f | T_\alpha \) for \( \emptyset \neq \alpha \subseteq [d] \) is continuous. Then
\[
\lim_{r \to \infty} f^{(r)}(x) = f(x) \quad \forall x \in [0,1]^d,
\]
i.e. the Bernstein polynomials converge pointwise to \( f \) everywhere.
For a function $f$ of $d$ variables we’ll use a short notation for its partial derivatives (if they exist). Let $p \in \mathbb{N}_0^d \setminus \{0\}$, then
\[
f_p := \frac{\partial^{|p|} f}{\partial x_1^{p_1} \cdots \partial x_d^{p_d}},
\]
complemented by $f_{0^d} := f$.

**Lemma 2.** Let $f : [0, 1]^d \to \mathbb{R}$ be arbitrary, $0 \neq p \in \mathbb{N}_0^d$.

(i) If $\Delta_h^p f \geq 0 \forall h \in \mathbb{R}_+^d$ then $(f^{(r)})_p \geq 0 \forall r \in \mathbb{N}$.

(ii) If $f$ is in addition $C^\infty$, then $f_p \geq 0$.

**Theorem 4.** Let $f : [0, 1]^d \to \mathbb{R}$ be a $C^\infty$-function, $n \in \mathbb{N}^d$, $k \in \mathbb{N}$.

Then

(i) $f$ is $n \uparrow \iff f_p \geq 0 \forall 0 \neq p \leq n, p \in \mathbb{N}_0^d$

(ii) $f$ is $k \uparrow \iff f_p \geq 0 \forall 0 < |p| \leq k, p \in \mathbb{N}_0^d$.

**Proof.** (i) "$\Rightarrow$": follows from Lemma 2.

"$\Leftarrow$": Let for $m \in \mathbb{N}$ $\sigma_m : \mathbb{R}^m \to \mathbb{R}$ be the sum function, $\sigma_n := \sigma_{n_1} \times \sigma_{n_2} \times \cdots \times \sigma_{n_d}$. By [12], Theorem 5 we have
\[
f is n \uparrow \iff f \circ \sigma_n is 1_{|n|} \uparrow on J := \prod_{i=1}^d \left[0, \frac{1}{n_i}\right]^{n_i}.
\]
The chain rule gives
\[
(f \circ \sigma_n)_{1_{|n|}} = f_n \circ \sigma_n \geq 0,
\]
so that for $x, x + h \in J, h \geq 0$ by Fubini’s theorem
\[
\left(\Delta_h^{1_{|n|}}(f \circ \sigma_n)\right)(x) = \int_{[x, x+h]} (f \circ \sigma_n)_{1_{|n|}} d\lambda^{|n|} \geq 0.
\]
\[\square\]
Examples 2. (a) \( f(x, y) := x^2 y - ax^2 y^2 + y^2 \) on \([0, 1]^2\), \( 0 < a \leq \frac{1}{2} \).
Since \( f_p \geq 0 \) for \( p \in \{(1,0), (0,1), (1,1), (2,0), (0,2)\} \), \( f \) is 2-\( \uparrow \);
but \( f_{(1,2)}(x, y) = -4ax \) shows that \( f \) is neither 3-\( \uparrow \) nor (2,2)-\( \uparrow \).

(b) \( f : \mathbb{R}_+^2 \to \mathbb{R} \) is defined by \( f(0, 0) := 0 \) and else
\[
f(x, y) := \frac{xy(x^2 - y^2)}{x^2 + y^2} + 13 \cdot (x^2 + y^2) + 3xy
\]
See [8], page 321, where it is given as an example of an ultramodular function on \( \mathbb{R}_+^2 \) (which doesn’t automatically include that it is increasing). However, all partial derivatives \( f_p \) with \( 0 < |p| \leq 2 \) are \( \geq 0 \), hence \( f \) is 2-\( \uparrow \) (and not 3-\( \uparrow \), BTW).

(c) With the abbreviation \( x^\alpha := \prod_{i \in \alpha} x_i \) for \( \alpha \subseteq [d], \) \( x^\emptyset := 1 \), a polynomial of the form
\[
f(x) = \sum_{\alpha \subseteq [d]} c_\alpha x^\alpha
\]
is called multilinear. \( f \) is affine in each variable, therefore \( f_p = 0 \) whenever \( p_i > 1 \) for some \( i \). Hence \( f \) is k-\( \uparrow \) iff \( f_p \geq 0 \)
\( \forall p \leq 1_d \) with \( 0 < |p| \leq k \), and \( n \)-\( \uparrow \) iff \( f \) is \((n \wedge 1_d) \)-\( \uparrow \). The example \( (d = 3) \)
\[
f(x) := x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3
\]
is thus 2-\( \uparrow \) on \([0,1]^3\), but not 3-\( \uparrow \), since \( f_{(1,1,1)} = -1 \). And \( f \)
is \((n, n, 0) \)-\( \uparrow \forall n \).

Theorem 5. Let \( f : [0, 1]^d \to \mathbb{R}, \) \( 2d \leq n \in \mathbb{N}_0^d, \) \( 2 \leq k \in \mathbb{N} \). The
Bernstein polynomials of \( f \) are denoted \( f^{(1)}, f^{(2)}, \ldots \).

(i) If \( f \) is \( n \)-\( \uparrow \) then so is each \( f^{(r)} \), and \( f^{(r)} \to f \) pointwise.

(ii) If \( f \) is \( k \)-\( \uparrow \) then so is each \( f^{(r)} \), and \( f^{(r)} \to f \) pointwise.
The main results

The proof of Theorem 6 below makes use of a far reaching generalization of the usual multivariate chain rule. This admirable result was shown by Constantine and Savits ([3], Theorem 2.1), and we present it here, keeping (almost) their notation.

Let $d, m \in \mathbb{N}$, let $g_1, \ldots, g_m$ be defined and $C^\infty$ in a neighborhood of $x^{(0)} \in \mathbb{R}^d$ (real-valued), put $g := (g_1, \ldots, g_m)$, let $f$ be defined and $C^\infty$ in a neighborhood of $y^{(0)} := g(x^{(0)}) \in \mathbb{R}^m$.

For $\mu, \nu \in \mathbb{N}_0^d$ define

(i) $|\mu| < |\nu|

or

$\mu < \nu \iff$ (ii) $|\mu| = |\nu|$ and $\mu_1 < \nu_1$

or

(iii) $|\mu| = |\nu|$, $\mu_1 = \nu_1, \ldots, \mu_k = \nu_k, \mu_{k+1} < \nu_{k+1}$, $\exists k \in [d-1]$

(implying $\mu \neq \nu$).

Examples:

(a) $(1, 3, 0, 4, 1) < (1, 3, 1, 1, 3)$, here $k = 2$
(b) $e_d < e_{d-1} < \cdots < e_1$
(c) For $d = 1$ we have $\mu < \nu \iff \mu < \nu$. 

We need some abbreviations:

\[ D_x^\nu := \frac{\partial |\nu|}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}} \text{ for } |\nu| > 0, \quad D_0^0 := \text{Id} \]

\[ x^\nu := \prod_{i=1}^d x_i^{\nu_i}, \quad \nu! := \prod_{i=1}^d \nu_i!, \quad |\nu| := \sum_{i=1}^d \nu_i \]

\[ g_{\mu}^{(i)} := (D_x^\mu g_i)(x^{(0)}), \quad g_{\mu} := (g_{\mu}^{(1)}, \ldots, g_{\mu}^{(m)}) \]

\[ f_{\lambda} := (D_y^\lambda f)(y^{(0)}) \]

\[ h := f \circ g, \quad h_{\nu} := (D_x^{\nu} h)(x^{(0)}) \]

and, for \( \nu \in \mathbb{N}_0^d, \lambda \in \mathbb{N}_0^m, s \in \mathbb{N}, s \leq |\nu| \)

\[ P_s(\nu, \lambda) := \left\{ (k_1, \ldots, k_s; l_1, \ldots, l_s) \mid |k_j| > 0, 0 < l_1 < \cdots < l_s, \sum_{j=1}^s k_j = \lambda, \sum_{j=1}^s l_j = |\nu| \right\} \]

where (of course) \( k_j \in \mathbb{N}_0^m \) and \( l_j \in \mathbb{N}_0^d \). (For some values of \( s \) these sets may be empty.)

The announced formula by Constantine and Savits then reads

\[ h_{\nu} = \sum_{1 \leq |\lambda| \leq |\nu|} f_{\lambda} \cdot \sum_{s=1}^{|\nu|} \nu! \cdot \prod_{j=1}^s \frac{(g_{l_j})^{k_j}}{(k_j!) \cdot (l_j!)^{k_j}} \quad (**) \]

This formula reduces for \( d = 1 \) to the classical one of Faa di Bruno from 1855, see [3].

One more result is needed, allowing general d.f.s to be "replaced" by \( C^\infty \) ones:

**Lemma 3.** \( (i) \) Let \( (\Omega, \mathcal{A}, \rho) \) be a finite measure space, \( \emptyset \neq \mathcal{B} \subseteq \mathcal{A} \) a finite collection of measurable sets. Then there is another finite measure \( \rho_0 \) on \( \mathcal{A} \) with finite support such that \( \rho_0|\mathcal{B} = \rho|\mathcal{B} \).
Let \( F \) on \( \mathbb{R}^d \) be the d.f. of some finite measure, \( \emptyset \neq B \subseteq \mathbb{R}^d \) a finite subset. Then there is a \( C^\infty \) d.f. \( \tilde{F} \) on \( \mathbb{R}^d \) such that \( \tilde{F}|B = F|B \).

**Theorem 6.** Let \( f : [0, 1]^m \to \mathbb{R}_+ \) be \( d \uparrow \) \((d \geq 2)\) and let \( g_1, \ldots, g_m : \mathbb{R}^d \to [0, 1] \) be d.f.s of (subprobability) measures on \( \mathbb{R}^d \). Then also \( f \circ (g_1, \ldots, g_m) \) is a d.f. on \( \mathbb{R}^d \).

**Proof.** Put \( g := (g_1, \ldots, g_m) : \mathbb{R}^d \to [0, 1]^m \), \( h := f \circ g \). By Lemma 1 also \( h \) is right-continuous, and it remains to show that \( h \) is \( 1_d \uparrow \), the crucial property of a d.f. on \( \mathbb{R}^d \).

A consequence of Theorem 5 is that we may assume \( f \) to be \( C^\infty \), and we first let also \( g_1, \ldots, g_m \) be \( C^\infty \) functions.

Now to the general case: in order to see that \( h = f \circ g \) is \( 1_d \uparrow \), we have to show for given \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}_+^d \)

\[
\left( \Delta^{1_d}_\xi h \right)(x) = h(x + \xi) \mp \cdots + (-1)^d h(x) \geq 0,
\]

(as well as the analogue for some variables fixed, which is shown similarly).

In Lemma 3 we choose the finite set

\[
\left\{ x + \sum_{i \in \alpha} \xi_i e_i \mid \alpha \subseteq [d] \right\} =: B
\]

and find \( C^\infty \) d.f.s \( \tilde{g}_1, \ldots, \tilde{g}_m \) such that \( \tilde{g}_i|B = g_i|B \) for each \( i \leq m \). Then

\[
0 \leq \left( \Delta^{1_d}_\xi (f \circ \tilde{g}) \right)(x) = \left( \Delta^{1_d}_\xi h \right)(x)
\]

thus finishing the proof. \( \square \)
Remark 4. If for a given $f$ the conclusion of Theorem 6 holds for all d.f.s $g_1, \ldots, g_m$, then $f$ must be $d\uparrow$. This follows from Theorem 1(v), since each component of an affine positive function $\varphi$ is of course $1_d\uparrow$.

Examples 3. (a) We saw before that $f(x) := x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$ is 2\uparrow on $[0, 1]^3$. Hence for arbitrary bivariate d.f.s $g_1, g_2, g_3$ also $g_1g_2 + g_1g_3 + g_2g_3 - g_1g_2g_3$ is a d.f., while $f$ itself is not a 3-dimensional d.f..

(b) Put $f_a(t) := (t - a) +/(1 - a)$ for $t \in [0, 1]$ and $a \in [0, 1]$, complemented by $f_1 := 1_{[1]}$. Then $\{f^n_\alpha | a \in [0, 1]\}$ are the "essential" extreme points for $(n+1)-\uparrow$ functions on $[0, 1]$, and $\{f^n_{a_1} \otimes \cdots \otimes f^n_{a_d} | a \in [0, 1]^d\}$ correspondingly for $(n+1_d)-\uparrow$ functions on $[0, 1]^d$, cf. [11]. In the bivariate case, $f_a \otimes f_b$ is $(2, 2)-\uparrow$, in particular 2\uparrow, so that $f_c \circ (f_a \otimes f_b)$ is 2\uparrow on $[0, 1]^2$. For any bivariate d.f.s $g_1, g_2$ we see that

$$\left[\frac{(g_1 - a)_+ \cdot (g_2 - b)_+}{(1 - a) \cdot (1 - b)} - c\right]_+ , \quad (a, b, c) \in [0, 1]^3$$

is again a bivariate d.f..

Another important property of $k\uparrow$ functions is their "universal" compatibility and composebility within their class, made precise in

Theorem 7. Let $m, d, k \in \mathbb{N}$, $J \subseteq \mathbb{R}^m$ and $I \subseteq \mathbb{R}^d$ be non-degenerate intervals, $g = (g_1, \ldots, g_m) : I \to J$, $f : J \to \mathbb{R}$, each $g_i$ and $f$ being $k\uparrow$. Then also $f \circ g$ is $k\uparrow$.

Proof. The case $k = 1$ being obvious, let’s assume $k \geq 2$. Since any non-degenerate interval is an increasing union of compact non-degenerate subintervals, we may choose $I = [0, 1]^d$ and $J = [0, 1]^m$. 

11
By Theorem 1 we have to show that \( h := f \circ g \) is \( n \uparrow \) for any \( n \in \mathbb{N}_0^d \) such that \( 0 < |n| \leq k \). Since the variables \( i \) with \( n_i = 0 \) do not enter, we may and do assume \( n \in \mathbb{N}^d \), in particular \( k \geq d \). Then each \( g_i \) is \( n \uparrow \), or equivalently, by [12] Theorem 5, \( g_i \circ \sigma_n \) is \( 1_{|n|} \uparrow \) on \( \prod_{i \leq d} \left[ 0, \frac{1}{n_i} \right]^{n_i} \). Theorem 6 above now implies that also

\[ f \circ (g_1 \circ \sigma_n, \ldots, g_m \circ \sigma_n) = h \circ \sigma_n \]

is \( 1_{|n|} \uparrow \), which in turn means that \( h \) is \( n \uparrow \). \( \Box \)

**An open problem**

While \( n \uparrow \) functions on \([0,1]^d\), non-negative and normalized, are a Bauer simplex, with "essentially" \( \{ f_{a_1} \otimes \cdots \otimes f_{a_d} \mid a \in [0,1]^d \} \) as their extreme points (see Examples 3(b) above), not much so far is known for \( k \uparrow \) functions. Let's consider \( d = k = 2 \) and

\[ K := \{ f : [0,1]^2 \to [0,1] \mid f \text{ is } 2 \uparrow \text{ and } f(1,1) = 1 \} \]

\( K \) is obviously convex and compact, and also stable under (pointwise) multiplication. It is easy to see that each \( f_c \circ (f_a \otimes f_b) \) is an extreme point of \( K \) - but that's it, for the time being.