Functions operating on several multivariate distribution functions

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Let $I_1, \ldots, I_d \subseteq \mathbb{R}$ be non-degenerate intervals, $I \coloneqq I_1 \times \cdots \times I_d$, and let $f : I \to \mathbb{R}$ be any function. For $s \in I, h \in \mathbb{R}_+^d$ such that also $s + h \in I$, put

$$(E_h f)(s) \coloneqq f(s+h)$$

and $\Delta_h \coloneqq E_h - E_0$, i.e. $(\Delta_h f)(s) \coloneqq f(s+h) - f(s)$. Since $\{E_h \,|\, h \in \mathbb{R}_+^d\}$ is commutative (where defined), so is also $\{\Delta_h \,|\, h \in \mathbb{R}_+^d\}$. In particular, with $e_1, \ldots, e_d$ denoting standard unit vectors in $\mathbb{R}^d$, $\Delta_{h_1 e_1}, \ldots, \Delta_{h_d e_d}$ commute. As usual, $\Delta_h^0 f \coloneqq f$ (also for $h = 0$, but clearly $\Delta_0 f = 0$). For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $h = (h_1, \ldots, h_d) \in \mathbb{R}_+^d$ we put

$$\Delta_h^{\mathbf{n}} \coloneqq \Delta_{h_1 e_1}^{n_1} \Delta_{h_2 e_2}^{n_2} \ldots \Delta_{h_d e_d}^{n_d},$$

so that $(\Delta_h^{\mathbf{n}} f)(s)$ is defined for $s, s + \sum_{i=1}^d n_i h_i e_i \in I$.

Definition. $f : I \to \mathbb{R}$ is $\mathbf{n}$ -$\uparrow$ (read "$\mathbf{n}$-*increasing*") iff $(\Delta_h^{\mathbf{p}} f)(s) \geq 0 \, \forall s \in I, h \in \mathbb{R}_+^d, \mathbf{p} \in \mathbb{N}_0^d, 0 \neq \mathbf{p} \leq \mathbf{n}$, such that $s_j + p_j h_j \in I_j \, \forall j \in [d]$.

A specially important case is $\mathbf{n} = \mathbf{1}_d$; being $\mathbf{1}_d$ -$\uparrow$ is the "crucial" property of d.f.s. More precisely: $f : I \to \mathbb{R}_+$ is the d.f. of a (non-negative) measure μ, i.e. $f(s) = \mu\left([-\infty, s] \cap \bar{I}\right)$ $\forall s \in I$, if and only if f is $\mathbf{1}_d$ -$\uparrow$ and right-continuous; c.f. [9], Theorem 7.

Let us for a moment consider the case $d = 1$. Then $I \subseteq \mathbb{R}, \mathbf{n} = n \in \mathbb{N}$, we assume $n \geq 2$, and a famous old result of Boas and Widder ([1], Lemma 1) shows that a continuous function $f : I \to \mathbb{R}$ is n -$\uparrow$ (i.e. $\Delta_h^j f \geq 0$ $\forall j \in [n]$, $\forall h > 0$) iff

$$\left(\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_j} f\right)(s) \geq 0$$

$\forall j \in [n], \forall h_1, \ldots, h_j > 0$ such that $s, s+h_1+\cdots+h_j \in I$. For $n = 2$, f is 2 - $\uparrow$ iff it is increasing and convex (and BTW automatically continuous on $I \setminus \{\sup I\}$).

The following definition now seems to be natural:

Definition. Let $I_1, \ldots, I_d \subseteq \mathbb{R}$ be non-degenerate intervals, $I = I_1 \times \cdots \times I_d$, $f : I \to \mathbb{R}$, and $k \in \mathbb{N}$. Then f is called *k-increasing* ("k-$\uparrow$") iff $\forall j \in [k]$, $\forall h^{(1)}, \ldots, h^{(j)} \in \mathbb{R}_+^d$, $\forall s \in I$ such that $s + h^{(1)} + \cdots + h^{(j)} \in I$

$$\left(\Delta_{h^{(1)}}\ldots\Delta_{h^{(j)}} f\right)(s) \geq 0.$$

(We do not assume f to be continuous.)

We mentioned already that a univariate f is 2-$\uparrow$ iff it is increasing and convex. But also multivariate 2-$\uparrow$ functions are well-known: they are called *ultramodular*, mostly ultramodular aggregation functions, the latter meaning they are also increasing, and defined as functions $f : [0,1]^d \to [0,1]$ with $f(\mathbf{0}_d) = 0$ and $f(\mathbf{1}_d) = 1$. Some simple properties of k -$\uparrow$ functions are shown first.

Lemma 1. *Let* $f : [0,1]^d \to \mathbb{R}$ *be* 2 -$\uparrow$. *Then*

(i) f *is continuous iff* f *is continuous in* $\mathbf{1}_d$.

(ii) f *is right-continuous and on* $[0,1[^d$ *continuous.*

Our first theorem will state some equivalent conditions for f to be k - $\uparrow$. An essential ingredient will be positive linear (or affine) mappings: a linear function $\psi : \mathbb{R}^m \to \mathbb{R}^d$ is called *positive* iff $\psi(\mathbb{R}^m_+) \subseteq \mathbb{R}^d_+$; and an affine $\varphi : \mathbb{R}^m \to \mathbb{R}^d$ is positive iff its "linear part" $\varphi - \varphi(0)$ is.

Theorem 1. *Let* $I \subseteq \mathbb{R}^d$ *be a non-degenerate interval,* $f : I \to \mathbb{R}, k, d \in \mathbb{N}$. *Then there are equivalent:*

(i) f *is* k -$\uparrow$

(ii) f *is* $\mathbf{n}$ -$\uparrow$ $\forall \mathbf{n} \in \mathbb{N}_0^d$ *with* $0 < |\mathbf{n}| \leq k$

(iii) $\forall m \in \mathbb{N}$, $\forall$ *non-degenerate interval* $J \subseteq \mathbb{R}^m$, $\forall$ *positive affine* $\varphi : \mathbb{R}^m \to \mathbb{R}^d$ *such that* $\varphi(J) \subseteq I$, *also* $f \circ \varphi$ *is* k -$\uparrow$

(iv) $\forall m$, J, φ *as before, and* $\forall \mathbf{n} \in \mathbb{N}_0^m$ *with* $0 < |\mathbf{n}| \leq k$ *the function* $f \circ \varphi$ *is* $\mathbf{n}$ -$\uparrow$

(v) $\forall m$, J, φ *as before, and* $\forall \mathbf{n} \in \{0,1\}^m$ *with* $0 < |\mathbf{n}| \leq k$ *the function* $f \circ \varphi$ *is* $\mathbf{n}$ -$\uparrow$.

Corollary 1. *Let* $I \subseteq \mathbb{R}^d$ *and* $B \subseteq \mathbb{R}$ *be non-degenerate intervals. If* $g : I \to B$ *and* $f : B \to \mathbb{R}$ *are both* k -$\uparrow$, *then so is* $f \circ g$.

Theorem 2. *Let* $I \subseteq \mathbb{R}^{d_1}$ *and* $J \subseteq \mathbb{R}^{d_2}$ *be non-degenerate intervals,* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$, *both non-negative and* k -$\uparrow$. *Then also* $f \otimes g$ *is* k -$\uparrow$ *on* $I \times J$, *and in case* $I = J$ *the product* $f \cdot g$ *is* k -$\uparrow$, *too.*

Proof. We first apply (ii) of Theorem 1. For $0 \neq (\mathbf{m},\mathbf{n}) \in \mathbb{N}_0^{d_1} \times \mathbb{N}_0^{d_2}$, $(x,y) \in I \times J$, $h^{(1)} \in \mathbb{R}_+^{d_1}$, $h^{(2)} \in \mathbb{R}_+^{d_2}$ we have

$$\left[\Delta^{\mathbf{m},\mathbf{n}}_{(h^{(1)},h^{(2)})}(f \otimes g)\right](x,y) = \left(\Delta^{\mathbf{m}}_{h^{(1)}} f\right)(x) \cdot \left(\Delta^{\mathbf{n}}_{h^{(2)}} g\right)(y)$$

and for $|(\mathbf{m},\mathbf{n})| = |\mathbf{m}| + |\mathbf{n}| \leq k$ both factors on the RHS are non-negative. Since $\mathbf{m} = 0$ or $\mathbf{n} = 0$ is allowed, only $(\mathbf{m},\mathbf{n}) \neq 0$ being required, we need in fact $f \geq 0$ and $g \geq 0$.

For $I = J$ (with $d_1 = d_2 =: d$) let $\varphi : \mathbb{R}^d \to \mathbb{R}^{2d}$ be given by $\varphi(x) \coloneqq (x,x)$, a positive linear map. Then $\varphi(I) \subseteq I \times I$, and by Theorem 1 (iii) $(f \otimes g) \circ \varphi = f \cdot g$ is also k -$\uparrow$. □

We see that any monomial $f(x) = \prod_{i=1}^d x_i^{n_i}$ ($n_i \in \mathbb{N}$) is k -$\uparrow$ on $\mathbb{R}_+^d$ for each $k \in \mathbb{N}$. If $c_i \in]0,\infty[$ then $\prod_{i=1}^d x_i^{c_i}$ is k -$\uparrow$ on $\mathbb{R}_+^d$ at least for $c_i \geq k-1$, $i = 1,\ldots,d$.

Examples 1. (a) For $a > 0$ the function $f(x,y) \coloneqq (xy - a)_+$ is 2 -$\uparrow$ on $\mathbb{R}_+^2$, since $t \mapsto (t-a)_+$ is 2 -$\uparrow$ on $\mathbb{R}_+$, by Corollary 1. In [11] on page 261 it was shown that f is not $(2,1)$ -$\uparrow$ (resp. $(1,2)$ -$\uparrow$), but it is of course $(1,1)$ -$\uparrow$, and so a bivariate d.f.. The tensor product $g(x,y) \coloneqq (x-a)_+ \cdot (y-b)_+$ is $(2,2)$ -$\uparrow$ $\forall a,b > 0$, hence certainly 2 -$\uparrow$, but not 3 -$\uparrow$ since $x \mapsto (x-a)_+$ is not.

Similarly $(xyz - a)_+^2$ is 3-$\uparrow$ on $\mathbb{R}_+^3$, for $a > 0$, and of course $(xy-a)_+^2$ is 3 -$\uparrow$ on $\mathbb{R}_+^2$. We'll see later on that $xy + xz + yz - xyz$ is 2 -$\uparrow$ on $[0,1]^3$, but not 3 -$\uparrow$.

(b) Consider $f_n(t) \coloneqq t^n/(1+t)$ for $t \geq 0$. It was shown in [6], Lemma 2.4, that f_n is n-$\uparrow$ (it is not $(n+1)$-$\uparrow$). So for any non-negative n-$\uparrow$ function g on any interval in any dimension, $g^n/(1+g)$ is n -$\uparrow$, too.

Approximation by Bernstein polynomials

The proof of our main result relies heavily on these special polynomials, since they inherit the monotonicity properties of interest. To define them we introduce for $r \in \mathbb{N}$, $i \in \{0, 1, \ldots, r\}$

$$b_{i,r}(t) \coloneqq \binom{r}{i} t^i (1-t)^{r-i}, \quad t \in \mathbb{R}$$

and for $\mathbf{i} = (i_1, \ldots, i_d) \in \{0, 1, \ldots, r\}^d$

$$B_{\mathbf{i},r} \coloneqq b_{i_1,r} \otimes \cdots \otimes b_{i_d,r}.$$

For any $f : [0,1]^d \to \mathbb{R}$ the associated Bernstein polynomials $f^{(1)}, f^{(2)}, \ldots$ are defined by

$$f^{(r)} \coloneqq \sum_{\mathbf{0}_d \leq \mathbf{i} \leq \mathbf{r}_d} f\left(\frac{\mathbf{i}}{r}\right) \cdot B_{\mathbf{i},r}.$$

It is perhaps not so well-known, that for each continuity point x of f we have

$$f^{(r)}(x) \to f(x), \quad r \to \infty.$$

In the following the "upper right boundary" of $[0,1]^d$ will play a role. Let for $\alpha \subseteq [d]$

$$T_\alpha \coloneqq \{x \in [0,1]^d \mid x_i < 1 \Leftrightarrow i \in \alpha\}.$$

Then $[0,1]^d = \dot\bigcup_{\alpha \subseteq [d]} T_\alpha$, $T_\emptyset = \{\mathbf{1}_d\}$ and $T_{[d]} = [0,1[^d$. The union $\bigcup_{\alpha \subsetneqq [d]} T_\alpha$ is called the *upper right boundary* of $[0,1]^d$.

Theorem 3. *Let $f : [0,1]^d \to \mathbb{R}$ have the property that each restriction $f|T_\alpha$ for $\emptyset \neq \alpha \subseteq [d]$ is continuous. Then*

$$\lim_{r \to \infty} f^{(r)}(x) = f(x) \quad \forall x \in [0,1]^d,$$

i.e. the Bernstein polynomials converge pointwise to f everywhere.

For a function f of d variables we'll use a short notation for its partial derivatives (if they exist). Let $\mathbf{p} \in \mathbb{N}_0^d \setminus \{0\}$, then

$$f_{\mathbf{p}} \coloneqq \frac{\partial^{|\mathbf{p}|} f}{\partial x_1^{p_1} \dots \partial x_d^{p_d}},$$

complemented by $f_{\mathbf{0}_d} \coloneqq f$.

Lemma 2. *Let $f : [0,1]^d \to \mathbb{R}$ be arbitrary, $0 \neq \mathbf{p} \in \mathbb{N}_0^d$.*

(i) If $\Delta_h^{\mathbf{p}} f \geq 0 \ \forall h \in \mathbb{R}_+^d$ then $\left(f^{(r)}\right)_{\mathbf{p}} \geq 0 \ \forall r \in \mathbb{N}$.

(ii) If f is in addition C^∞, then $f_{\mathbf{p}} \geq 0$.

Theorem 4. *Let $f : [0,1]^d \to \mathbb{R}$ be a C^∞-function, $\mathbf{n} \in \mathbb{N}^d$, $k \in \mathbb{N}$. Then*

(i) f is $\mathbf{n}$ -↑ $\Leftrightarrow$ $f_{\mathbf{p}} \geq 0 \ \forall 0 \neq \mathbf{p} \leq \mathbf{n}, \mathbf{p} \in \mathbb{N}_0^d$

(ii) f is k -↑ $\Leftrightarrow$ $f_{\mathbf{p}} \geq 0 \ \forall 0 < |\mathbf{p}| \leq k$, $\mathbf{p} \in \mathbb{N}_0^d$.

Proof. (i) "$\Rightarrow$": follows from Lemma 2.

"$\Leftarrow$": Let for $m \in \mathbb{N}$ $\sigma_m : \mathbb{R}^m \to \mathbb{R}$ be the sum function, $\sigma_{\mathbf{n}} \coloneqq \sigma_{n_1} \times \sigma_{n_2} \times \dots \times \sigma_{n_d}$. By [12], Theorem 5 we have

$$f \text{ is } \mathbf{n}\text{-}\uparrow \Leftrightarrow f \circ \sigma_{\mathbf{n}} \text{ is } \mathbf{1}_{|\mathbf{n}|}\text{-}\uparrow \text{ on } J \coloneqq \prod_{i=1}^{d} \left[0, \frac{1}{n_i}\right]^{n_i}.$$

The chain rule gives

$$(f \circ \sigma_{\mathbf{n}})_{\mathbf{1}_{|\mathbf{n}|}} = f_{\mathbf{n}} \circ \sigma_{\mathbf{n}} \geq 0,$$

so that for $x, x+h \in J, h \geq 0$ by Fubini's theorem

$$\left(\Delta_h^{\mathbf{1}_{|\mathbf{n}|}} (f \circ \sigma_{\mathbf{n}})\right)(x) = \int_{[x,x+h]} (f \circ \sigma_{\mathbf{n}})_{\mathbf{1}_{|\mathbf{n}|}} \, d\boldsymbol{\lambda}^{|\mathbf{n}|} \geq 0.$$

□

Examples 2. (a) $f(x,y) \coloneqq x^2y - ax^2y^2 + y^2$ on $[0,1]^2$, $0 < a \le \frac{1}{2}$. Since $f_{\mathbf{p}} \ge 0$ for $\mathbf{p} \in \{(1,0),(0,1),(1,1),(2,0),(0,2)\}$, f is 2-$\uparrow$; but $f_{(1,2)}(x,y) = -4ax$ shows that f is neither 3-$\uparrow$ nor $(2,2)$-$\uparrow$.

(b) $f : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by $f(0,0) \coloneqq 0$ and else

$$f(x,y) \coloneqq \frac{xy(x^2-y^2)}{x^2+y^2} + 13\cdot(x^2+y^2) + 3xy$$

See [8], page 321, where it is given as an example of an ultramodular function on $\mathbb{R}^2_+$ (which doesn't automatically include that it is increasing). However, all partial derivatives $f_{\mathbf{p}}$ with $0 < |\mathbf{p}| \le 2$ are ≥ 0, hence f is 2-$\uparrow$ (and not 3-$\uparrow$, BTW).

(c) With the abbreviation $x^\alpha \coloneqq \prod_{i\in\alpha} x_i$ for $\alpha \subseteq [d]$, $x^\emptyset \coloneqq 1$, a polynomial of the form

$$f(x) = \sum_{\alpha\subseteq[d]} c_\alpha x^\alpha$$

is called *multilinear*. f is affine in each variable, therefore $f_{\mathbf{p}} = 0$ whenever $p_i > 1$ for some i. Hence f is k-$\uparrow$ iff $f_{\mathbf{p}} \ge 0$ $\forall \mathbf{p} \le \mathbf{1}_d$ with $0 < |\mathbf{p}| \le k$, and $\mathbf{n}$-$\uparrow$ iff f is $(\mathbf{n} \wedge \mathbf{1}_d)$-$\uparrow$. The example ($d = 3$)

$$f(x) \coloneqq x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

is thus 2-$\uparrow$ on $[0,1]^3$, but not 3-$\uparrow$, since $f_{(1,1,1)} = -1$. And f is $(n,n,0)$-$\uparrow$ $\forall n$.

Theorem 5. *Let $f : [0,1]^d \to \mathbb{R}$, $\mathbf{2}_d \le \mathbf{n} \in \mathbb{N}_0^d$, $2 \le k \in \mathbb{N}$. The Bernstein polynomials of f are denoted $f^{(1)}, f^{(2)}, \ldots$.*

(i) If f is $\mathbf{n}$-$\uparrow$ then so is each $f^{(r)}$, and $f^{(r)} \to f$ pointwise.

(ii) If f is k-$\uparrow$ then so is each $f^{(r)}$, and $f^{(r)} \to f$ pointwise.

The main results

The proof of Theorem 6 below makes use of a far reaching generalization of the usual multivariate chain rule. This admirable result was shown by Constantine and Savits ([3], Theorem 2.1), and we present it here, keeping (almost) their notation.

Let $d, m \in \mathbb{N}$, let $g_1, \ldots, g_m$ be defined and C^∞ in a neighborhood of $x^{(0)} \in \mathbb{R}^d$ (real-valued), put $g \coloneqq (g_1, \ldots, g_m)$, let f be defined and C^∞ in a neighborhood of $y^{(0)} \coloneqq g(x^{(0)}) \in \mathbb{R}^m$.

For $\mu, \nu \in \mathbb{N}_0^d$ define

$$\mu \prec \nu \quad :\Leftrightarrow \quad \begin{array}{l} \text{(i) } |\mu| < |\nu| \\ \text{or} \\ \text{(ii) } |\mu| = |\nu| \text{ and } \mu_1 < \nu_1 \\ \text{or} \\ \text{(iii) } |\mu| = |\nu|, \mu_1 = \nu_1, \ldots, \mu_k = \nu_k, \mu_{k+1} < \nu_{k+1}, \exists k \in [d-1] \end{array}$$

(implying $\mu \neq \nu$).

<u>Examples:</u>

(a) $(1,3,0,4,1) \prec (1,3,1,1,3)$, here $k = 2$
(b) $e_d \prec e_{d-1} \prec \cdots \prec e_1$
(c) For $d = 1$ we have $\mu \prec \nu \Leftrightarrow \mu < \nu$.

We need some abbreviations:

$$
\begin{aligned}
D_x^\nu &\coloneqq \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\dots\partial x_d^{\nu_d}} \text{ for } |\nu|>0, \quad D_x^0 \coloneqq \mathrm{Id}\\
x^\nu &\coloneqq \prod_{i=1}^d x_i^{\nu_i}, \quad \nu! \coloneqq \prod_{i=1}^d \nu_i!, \quad |\nu| \coloneqq \sum_{i=1}^d \nu_i\\
g_\mu^{(i)} &\coloneqq \left(D_x^\mu g_i\right)(x^{(0)}), \quad g_\mu \coloneqq \left(g_\mu^{(1)},\dots,g_\mu^{(m)}\right)\\
f_\lambda &\coloneqq \left(D_y^\lambda f\right)(y^{(0)})\\
h &\coloneqq f\circ g, \quad h_\nu \coloneqq \left(D_x^\nu h\right)(x^{(0)})
\end{aligned}
$$

and, for $\nu\in\mathbb{N}_0^d, \lambda\in\mathbb{N}_0^m, s\in\mathbb{N}, s\le|\nu|$

$$P_s(\nu,\lambda) \coloneqq \Bigg\{ (k_1,\dots,k_s;l_1,\dots,l_s) \,\big|\, |k_j|>0, 0\prec l_1 \prec \dots \prec l_s, \sum_{j=1}^s k_j = \lambda, \sum_{j=}^{s}$$

where (of course) $k_j\in\mathbb{N}_0^m$ and $l_j\in\mathbb{N}_0^d$. (For some values of s these sets may be empty.)

The announced formula by Constantine and Savits then reads

$$h_\nu = \sum_{1\le|\lambda|\le|\nu|} f_\lambda \cdot \sum_{s=1}^{|\nu|} \sum_{P_s(\nu,\lambda)} \nu! \cdot \prod_{j=1}^s \frac{(g_{l_j})^{k_j}}{(k_j!)\cdot(l_j!)^{|k_j|}} \qquad (**)$$

This formula reduces for $d=1$ to the classical one of Faa di Bruno from 1855, see [3].

One more result is needed, allowing general d.f.s to be "replaced" by C^∞ ones:

Lemma 3. *(i) Let $(\Omega,\mathcal{A},\rho)$ be a finite measure space, $\emptyset\neq\mathcal{B}\subseteq\mathcal{A}$ a finite collection of measurable sets. Then there is another finite measure ρ_0 on $\mathcal{A}$ with finite support such that $\rho_0|\mathcal{B}=\rho|\mathcal{B}$.*

(ii) Let F on $\mathbb{R}^d$ be the d.f. of some finite measure, $\emptyset \neq B \subseteq \mathbb{R}^d$ a finite subset. Then there is a C^∞ d.f. $\tilde{F}$ on $\mathbb{R}^d$ such that $\tilde{F}|B = F|B$.

Theorem 6. *Let $f : [0,1]^m \to \mathbb{R}_+$ be d-$\uparrow$ ($d \geq 2$) and let $g_1, \dots, g_m : \mathbb{R}^d \to [0,1]$ be d.f.s of (subprobability) measures on $\mathbb{R}^d$. Then also $f \circ (g_1, \dots, g_m)$ is a d.f. on $\mathbb{R}^d$.*

Proof. Put $g := (g_1, \dots, g_m) : \mathbb{R}^d \to [0,1]^m$, $h := f \circ g$. By Lemma 1 also h is right-continuous, and it remains to show that h is $\mathbf{1}_d$-$\uparrow$, the crucial property of a d.f. on $\mathbb{R}^d$.

A consequence of Theorem 5 is that we may assume f to be C^∞, and we first let also $g_1, \dots, g_m$ be C^∞ functions.

Now to the general case: in order to see that $h = f \circ g$ is $\mathbf{1}_d$-$\uparrow$, we have to show for given $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d_+$

$$\left(\Delta_\xi^{\mathbf{1}_d} h\right)(x) = h(x+\xi) \mp \cdots + (-1)^d h(x) \geq 0,$$

(as well as the analogue for some variables fixed, which is shown similarly).

In Lemma 3 we choose the finite set

$$\left\{ x + \sum_{i \in \alpha} \xi_i e_i \,|\, \alpha \subseteq [d] \right\} =: B$$

and find C^∞ d.f.s $\tilde{g}_1, \dots, \tilde{g}_m$ such that $\tilde{g}_i|B = g_i|B$ for each $i \leq m$. Then

$$0 \leq \left(\Delta_\xi^{\mathbf{1}_d}(f \circ \tilde{g})\right)(x) = \left(\Delta_\xi^{\mathbf{1}_d} h\right)(x)$$

thus finishing the proof. □

Remark 4. If for a given f the conclusion of Theorem 6 holds for all d.f.s $g_1, \ldots, g_m$, then f must be d-$\uparrow$. This follows from Theorem 1(v), since each component of an affine positive function φ is of course $\mathbf{1}_d$-$\uparrow$.

Examples 3. (a) We saw before that $f(x) \coloneqq x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$ is 2 - $\uparrow$ on $[0,1]^3$. Hence for arbitrary bivariate d.f.s g_1, g_2, g_3 also $g_1g_2 + g_1g_3 + g_2g_3 - g_1g_2g_3$ is a d.f., while f itself is not a 3-dimensional d.f..

(b) Put $f_a(t) \coloneqq (t-a)_+/(1-a)$ for $t \in [0,1]$ and $a \in [0,1[$, complemented by $f_1 \coloneqq 1_{\{1\}}$. Then $\{f_\alpha^n \,|\, a \in [0,1]\}$ are the "essential" extreme points for $(n+1)$-$\uparrow$ functions on $[0,1]$, and $\{f_{a_1}^{n_1} \otimes \cdots \otimes f_{a_d}^{n_d} \,|\, a \in [0,1]^d\}$ correspondingly for $(\mathbf{n} + \mathbf{1}_d)$-$\uparrow$ functions on $[0,1]^d$, cf. [11]. In the bivariate case, $f_a \otimes f_b$ is $(2,2)$-$\uparrow$, in particular 2-$\uparrow$, so that $f_c \circ (f_a \otimes f_b)$ is 2-$\uparrow$ on $[0,1]^2$. For any bivariate d.f.s g_1, g_2 we see that

$$\left[\frac{(g_1 - a)_+ \cdot (g_2 - b)_+}{(1-a)\cdot(1-b)} - c\right]_+, \qquad (a,b,c) \in [0,1[^3$$

is again a bivariate d.f..

Another important property of k - $\uparrow$ functions is their "universal" compatibility and composebility within their class, made precise in

Theorem 7. *Let $m, d, k \in \mathbb{N}$, $J \subseteq \mathbb{R}^m$ and $I \subseteq \mathbb{R}^d$ be non-degenerate intervals, $g = (g_1, \ldots, g_m) : I \to J$, $f : J \to \mathbb{R}$, each g_i and f being k-$\uparrow$. Then also $f \circ g$ is k-$\uparrow$.*

Proof. The case $k = 1$ being obvious, let's assume $k \geq 2$. Since any non-degenerate interval is an increasing union of compact non-degenerate subintervals, we may choose $I = [0,1]^d$ and $J = [0,1]^m$.

By Theorem 1 we have to show that $h \coloneqq f \circ g$ is $\mathbf{n}$-$\uparrow$ for any $\mathbf{n} \in \mathbb{N}_0^d$ such that $0 < |\mathbf{n}| \leq k$. Since the variables i with $n_i = 0$ do not enter, we may and do assume $\mathbf{n} \in \mathbb{N}^d$, in particular $k \geq d$. Then each g_i is $\mathbf{n}$ -$\uparrow$, or equivalently, by [12] Theorem 5, $g_i \circ \sigma_{\mathbf{n}}$ is $\mathbf{1}_{|\mathbf{n}|}$ -$\uparrow$ on $\prod_{i \leq d} \left[0, \frac{1}{n_i}\right]^{n_i}$. Theorem 6 above now implies that also

$$f \circ (g_1 \circ \sigma_{\mathbf{n}}, \ldots, g_m \circ \sigma_{\mathbf{n}}) = h \circ \sigma_{\mathbf{n}}$$

is $\mathbf{1}_{|\mathbf{n}|}$ -$\uparrow$, which in turn means that h is $\mathbf{n}$ -$\uparrow$. $\square$

An open problem

While $\mathbf{n}$ -$\uparrow$ functions on $[0,1]^d$, non-negative and normalized, are a Bauer simplex, with "essentially" $\{f_{a_1} \otimes \cdots \otimes f_{a_d} \mid a \in [0,1]^d\}$ as their extreme points (see Examples 3(b) above), not much so far is known for k -$\uparrow$ functions. Let's consider $d = k = 2$ and

$$K \coloneqq \{f : [0,1]^2 \to [0,1] \mid f \text{ is } 2\text{-}\uparrow \text{ and } f(1,1) = 1\}.$$

K is obviously convex and compact, and also stable under (pointwise) multiplication. It is easy to see that each $f_c \circ (f_a \otimes f_b)$ is an extreme point of K - but that's it, for the time being.