Grothendieck compactness principle for the absolute weak topology

Vinícius Miranda[1]

Federal University of Uberlandia, Brazil

Joint work with Geraldo Botelho and José Lucas P. Luiz

Positivity XI
Ljubljana, Slovenia

[1]Supported by CNPq Grant 150894/2022-8

Theorem (Grothendieck, 1955)

Every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence.

Theorem (Grothendieck, 1955)

Every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence.

Example: The closed unit ball of ℓ_2, considered as a subset of c_0, is not contained in the closed convex hull of a weakly null sequence in c_0. (E. Odell and Y. Sternfeld, 1981).

Introduction and Background

Theorem (Grothendieck, 1955)

Every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence.

Example: The closed unit ball of ℓ_2, considered as a subset of c_0, is not contained in the closed convex hull of a weakly null sequence in c_0. (E. Odell and Y. Sternfeld, 1981).

Theorem (P. N. Dowling et al, 2012)

Every weakly compact subset of a Banach space is contained in the closed convex hull of a weakly null sequence if and only if the Banach space has the Schur property.

P. N. Dowling, D. Freeman, C.J. Lennard, E. Odell, B. Randrianantoanina and B. Turett, *A weak Grothendieck compactness principle*, J. Funct. Anal. 263(5) (2012) 1378-1381.

As expected, Grothendieck compactness-type principles have been considered for different topologies...

As expected, Grothendieck compactness-type principles have been considered for different topologies...

P. N. Dowling and D. Mupasiri, *A Grothendieck compactness principle for the Mackey dual topology*, J. Math. Anal. Appl. 410 (2014), 483-486.

K. Beanland and R. Causey, *A ξ-weak Grothendieck compactness principle*, Math. Proc. Cambridge Philos. Soc. 172(1) (2022), 231-246.

In Banach lattice theory, a well known topology is the absolute weak topology $|\sigma|(E, E^*)$ on a Banach lattice E, which lies between the weak and norm topologies, that is

$$w(E, E^*) \subset |\sigma|(E, E^*) \subset \tau_{\|\cdot\|}(E).$$

In Banach lattice theory, a well known topology is the absolute weak topology $|\sigma|(E, E^*)$ on a Banach lattice E, which lies between the weak and norm topologies, that is

$$w(E, E^*) \subset |\sigma|(E, E^*) \subset \tau_{\|\cdot\|}(E).$$

- Let us recall, that for a Banach lattice E, the absolute weak topology $|\sigma|(E, E^*)$ is the locally convex-solid topology on E generated by the family $\{p_{x^*} : x^* \in A\}$ of lattice seminorms, where $p_{x^*}(x) = |x^*|(|x|)$ for all $x \in E$ and $x^* \in E^*$.

In Banach lattice theory, a well known topology is the absolute weak topology $|\sigma|(E, E^*)$ on a Banach lattice E, which lies between the weak and norm topologies, that is

$$w(E, E^*) \subset |\sigma|(E, E^*) \subset \tau_{\|\cdot\|}(E).$$

- Let us recall, that for a Banach lattice E, the absolute weak topology $|\sigma|(E, E^*)$ is the locally convex-solid topology on E generated by the family $\{p_{x^*} : x^* \in A\}$ of lattice seminorms, where $p_{x^*}(x) = |x^*|(|x|)$ for all $x \in E$ and $x^* \in E^*$.
- It should be noted that a net $(x_\alpha)_\alpha$ in E is absolutely weakly null, i.e. $x_\alpha \overset{|\sigma|(E,E^*)}{\longrightarrow} 0$, if and only if $|x_\alpha| \overset{\omega}{\to} 0$.

In order to replace the weak topology in the weak Grothendieck's compactness principle (Dowling et al, 2012), a natural candidate to replace the Schur property is the positive Schur property...

In order to replace the weak topology in the weak Grothendieck's compactness principle (Dowling et al, 2012), a natural candidate to replace the Schur property is the positive Schur property...

- A Banach space X is said to have the Schur property if every weakly null sequence in X is norm null, that is

$$x_n \stackrel{\omega}{\rightarrow} 0 \text{ in } X \Rightarrow \|x_n\| \rightarrow 0.$$

In order to replace the weak topology in the weak Grothendieck's compactness principle (Dowling et al, 2012), a natural candidate to replace the Schur property is the positive Schur property...

- A Banach space X is said to have the Schur property if every weakly null sequence in X is norm null, that is

$$x_n \overset{\omega}{\to} 0 \text{ in } X \Rightarrow \|x_n\| \to 0.$$

- A Banach lattice E is said to have the positive Schur property if every positive weakly null sequence in E is norm null, that is

$$0 \leq x_n \overset{\omega}{\to} 0 \text{ in } E \Rightarrow \|x_n\| \to 0.$$

In order to replace the weak topology in the weak Grothendieck's compactness principle (Dowling et al, 2012), a natural candidate to replace the Schur property is the positive Schur property...

- A Banach space X is said to have the Schur property if every weakly null sequence in X is norm null, that is

$$x_n \overset{\omega}{\to} 0 \text{ in } X \Rightarrow \|x_n\| \to 0.$$

- A Banach lattice E is said to have the positive Schur property if every positive weakly null sequence in E is norm null, that is

$$0 \leq x_n \overset{\omega}{\to} 0 \text{ in } E \Rightarrow \|x_n\| \to 0.$$

  Or equivalently, if every absolutely weakly null sequence in E is norm null, that is

$$x_n \overset{|\sigma|(E,E^*)}{\longrightarrow} 0 \text{ in } E \Rightarrow \|x_n\| \to 0.$$

Conjecture

Every absolutely weakly compact subset of a Banach lattice is contained in the closed convex hull of an absolutely weakly null sequence if and only if the Banach lattice has the positive Schur property.

Conjecture

Every absolutely weakly compact subset of a Banach lattice is contained in the closed convex hull of an absolutely weakly null sequence if and only if the Banach lattice has the positive Schur property.

In our way to prove this conjecture, we realized that we would need a version of the well known Eberlein-Smulian theorem for the absolutely weak topology...

Theorem

1. (Smulian) Every weakly compact subset of a Banach space is sequentially weakly compact.

Theorem

1. (Smulian) Every weakly compact subset of a Banach space is sequentially weakly compact.
2. (Eberlein) Every sequentially weakly compact subset of a Banach space is weakly compact.

Theorem 1

Absolutely weakly compact subsets of Banach lattices are absolutely weakly sequentially compact.

Theorem 1

Absolutely weakly compact subsets of Banach lattices are absolutely weakly sequentially compact.

As a first application we give an example of a weakly compact set which is neither absolutely weakly compact nor absolutely weakly sequentially compact:

Theorem 1

Absolutely weakly compact subsets of Banach lattices are absolutely weakly sequentially compact.

As a first application we give an example of a weakly compact set which is neither absolutely weakly compact nor absolutely weakly sequentially compact:

Example: Letting $K := \{r_n : n \in \mathbb{N}\} \cup \{0\}$, where $(r_n)_n$ denotes the Rademcher's sequence in $L_1[0,1]$, we have that K is a weakly compact subset of $L_1[0,1]$. Nevertheless, K is not absolutely weakly sequentially compact, and so it can not be absolutely weakly compact.

Since the usual proof of the Eberlein part of the Eberlein-Šmulian Theorem uses the weak* compactness of the closed unit ball of the dual of any Banach space (Alaoglu's Theorem), we considered the absolute weak* topology $|\sigma|(E^*, E)$ on the dual E^* of a Banach lattice, which is the locally convex-solid topology on E^* generated by the family $\{q_x : x \in E\}$ of lattice seminorms, where $q_x(x^*) = |x^*|(|x|)$ for all $x^* \in E^*$ and $x \in E$. In particular, we have that

$$\sigma(E^*, E) \subset |\sigma|(E^*, E) \subset |\sigma|(E^*, E^{**}).$$

Since the usual proof of the Eberlein part of the Eberlein-Šmulian Theorem uses the weak* compactness of the closed unit ball of the dual of any Banach space (Alaoglu's Theorem), we considered the absolute weak* topology $|\sigma|(E^*, E)$ on the dual E^* of a Banach lattice, which is the locally convex-solid topology on E^* generated by the family $\{q_x : x \in E\}$ of lattice seminorms, where $q_x(x^*) = |x^*|(|x|)$ for all $x^* \in E^*$ and $x \in E$. In particular, we have that

$$\sigma(E^*, E) \subset |\sigma|(E^*, E) \subset |\sigma|(E^*, E^{**}).$$

It happens, however, that, in general, B_{E^*} is not absolutely weak* compact.

Since the usual proof of the Eberlein part of the Eberlein-Šmulian Theorem uses the weak* compactness of the closed unit ball of the dual of any Banach space (Alaoglu's Theorem), we considered the absolute weak* topology $|\sigma|(E^*, E)$ on the dual E^* of a Banach lattice, which is the locally convex-solid topology on E^* generated by the family $\{q_x : x \in E\}$ of lattice seminorms, where $q_x(x^*) = |x^*|(|x|)$ for all $x^* \in E^*$ and $x \in E$. In particular, we have that

$$\sigma(E^*, E) \subset |\sigma|(E^*, E) \subset |\sigma|(E^*, E^{**}).$$

It happens, however, that, in general, B_{E^*} is not absolutely weak* compact.

Proposition

If B_{E^*} is absolutely weak* compact, then E has order continuous norm.

Theorem 2

Let K be an absolutely weakly sequentially compact subset of a Banach lattice E. If E is separable or $B_{E^{**}}$ is absolutely weak* compact, then K is absolutely weakly compact.

Theorem 2

Let K be an absolutely weakly sequentially compact subset of a Banach lattice E. If E is separable or $B_{E^{**}}$ is absolutely weak* compact, then K is absolutely weakly compact.

To establish the usefulness of the theorem above in the nonseparable case, we should give examples of nonseparable Banach lattices E for which $B_{E^{**}}$ is absolutely weak* compact.

Theorem 2

Let K be an absolutely weakly sequentially compact subset of a Banach lattice E. If E is separable or $B_{E^{**}}$ is absolutely weak* compact, then K is absolutely weakly compact.

To establish the usefulness of the theorem above in the nonseparable case, we should give examples of nonseparable Banach lattices E for which $B_{E^{**}}$ is absolutely weak* compact.

Proposition

Let E be Banach lattice such that E^* and E^{**} have order continuous norms and E^{**} is atomic. Then $B_{E^{**}}$ is absolutely weak* compact. In particular, $B_{E^{**}} = B_E$ is absolutely weak* compact for every reflexive atomic Banach lattice.

Theorem 2

Let K be an absolutely weakly sequentially compact subset of a Banach lattice E. If E is separable or $B_{E^{**}}$ is absolutely weak* compact, then K is absolutely weakly compact.

To establish the usefulness of the theorem above in the nonseparable case, we should give examples of nonseparable Banach lattices E for which $B_{E^{**}}$ is absolutely weak* compact.

Proposition

Let E be Banach lattice such that E^* and E^{**} have order continuous norms and E^{**} is atomic. Then $B_{E^{**}}$ is absolutely weak* compact. In particular, $B_{E^{**}} = B_E$ is absolutely weak* compact for every reflexive atomic Banach lattice.

Example: Let Γ be an uncountable set and $1 < p < \infty$. Then $E = \ell_p(\Gamma)$ is a nonseparable reflexive atomic Banach lattice such that $B_E = B_{E^{**}}$ is absolutely weak* compact.

Main theorem

Theorem theorem

A Banach lattice E has the positive Schur property if and only if every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

Proof: $(\Rightarrow)$ Assume first that E has the positive Schur property and let K be an absolutely weakly compact subset of E.

Proof: $(\Rightarrow)$ Assume first that E has the positive Schur property and let K be an absolutely weakly compact subset of E.

- Since absolutely weakly compact sets are sequentially absolutely weakly compact (Theorem 1), every sequence $(x_n)_n \subset K$ has an absolutely weakly convergent subsequence $(x_{n_k})_k$, that is

$$x_{n_k} \overset{|\sigma|(E,E^*)}{\longrightarrow} x \in K.$$

Proof: $(\Rightarrow)$ Assume first that E has the positive Schur property and let K be an absolutely weakly compact subset of E.

- Since absolutely weakly compact sets are sequentially absolutely weakly compact (Theorem 1), every sequence $(x_n)_n \subset K$ has an absolutely weakly convergent subsequence $(x_{n_k})_k$, that is
$$x_{n_k} \overset{|\sigma|(E,E^*)}{\longrightarrow} x \in K.$$
- So $(x_{n_k} - x)_k$ is an absolutely weakly null sequence, and therefore $\|x_{n_k} - x\| \to 0$.

Proof: ($\Rightarrow$) Assume first that E has the positive Schur property and let K be an absolutely weakly compact subset of E.

- Since absolutely weakly compact sets are sequentially absolutely weakly compact (Theorem 1), every sequence $(x_n)_n \subset K$ has an absolutely weakly convergent subsequence $(x_{n_k})_k$, that is

$$x_{n_k} \overset{|\sigma|(E,E^*)}{\longrightarrow} x \in K.$$

- So $(x_{n_k} - x)_k$ is an absolutely weakly null sequence, and therefore $\|x_{n_k} - x\| \to 0$.

- Consequently, K is a norm compact subset of E, and by the Grothendieck's compactness principle K is contained in the closed convex hull of a norm null sequence.

Proof: $(\Rightarrow)$ Assume first that E has the positive Schur property and let K be an absolutely weakly compact subset of E.

- Since absolutely weakly compact sets are sequentially absolutely weakly compact (Theorem 1), every sequence $(x_n)_n \subset K$ has an absolutely weakly convergent subsequence $(x_{n_k})_k$, that is

$$x_{n_k} \overset{|\sigma|(E,E^*)}{\longrightarrow} x \in K.$$

- So $(x_{n_k} - x)_k$ is an absolutely weakly null sequence, and therefore $\|x_{n_k} - x\| \to 0$.
- Consequently, K is a norm compact subset of E, and by the Grothendieck's compactness principle K is contained in the closed convex hull of a norm null sequence.
- As norm null sequences are absolutely weakly null, we get this implication.

To prove the other implication, we will need following lemma:

Lemma 1

In a Banach space X one cannot find weakly null sequences $(x_n)_n$ and $(y_n)_n$ with $\|x_n\| = 1$ for every $n \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{\infty} \left[(n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_X \right] \subset \overline{\text{co}}(\{y_n : n \in \mathbb{N}\}).$$

To prove the other implication, we will need following lemma:

Lemma 1

In a Banach space X one cannot find weakly null sequences $(x_n)_n$ and $(y_n)_n$ with $\|x_n\| = 1$ for every $n \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{\infty} \left[(n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_X \right] \subset \overline{\text{co}}(\{y_n : n \in \mathbb{N}\}).$$

The proof of this lemma can be found within the proof of the weak Grothendieck's compactness principle (Dowling et al). Actually the fact stated in the lemma above is one of the most difficult parts of the proofs of this result, and fortunately, it can also be used in our case.

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

- If E fails to have the positive Schur property, there exists an absolutely weakly null sequence $(x_n)_n$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

- If E fails to have the positive Schur property, there exists an absolutely weakly null sequence $(x_n)_n$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$.
- Since $x_n \overset{|\sigma|(E,E^*)}{\longrightarrow} 0$, an standard diagonal argument shows that $C = \overline{co}(\{x_n : n \in \mathbb{N}\})$ is sequentially absolutely weakly compact.

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

- If E fails to have the positive Schur property, there exists an absolutely weakly null sequence $(x_n)_n$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$.
- Since $x_n \overset{|\sigma|(E,E^*)}{\longrightarrow} 0$, an standard diagonal argument shows that $C = \overline{co}(\{x_n : n \in \mathbb{N}\})$ is sequentially absolutely weakly compact. By considering the separable vector subspace $[x_n : n \in \mathbb{N}]$, we may find a separable Banach sublattice F of E containing $[x_n : n \in \mathbb{N}]$.

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

- If E fails to have the positive Schur property, there exists an absolutely weakly null sequence $(x_n)_n$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$.
- Since $x_n \overset{|\sigma|(E,E^*)}{\longrightarrow} 0$, an standard diagonal argument shows that $C = \overline{co}(\{x_n : n \in \mathbb{N}\})$ is sequentially absolutely weakly compact. By considering the separable vector subspace $[x_n : n \in \mathbb{N}]$, we may find a separable Banach sublattice F of E containing $[x_n : n \in \mathbb{N}]$.
- In particular, C is a sequentially absolutely weakly compact subset of F, and by Theorem 2, we obtain that C is absolutely weakly compact in F,

($\Leftarrow$) Now assume that every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.

- If E fails to have the positive Schur property, there exists an absolutely weakly null sequence $(x_n)_n$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$.
- Since $x_n \overset{|\sigma|(E,E^*)}{\longrightarrow} 0$, an standard diagonal argument shows that $C = \overline{co}(\{x_n : n \in \mathbb{N}\})$ is sequentially absolutely weakly compact. By considering the separable vector subspace $[x_n : n \in \mathbb{N}]$, we may find a separable Banach sublattice F of E containing $[x_n : n \in \mathbb{N}]$.
- In particular, C is a sequentially absolutely weakly compact subset of F, and by Theorem 2, we obtain that C is absolutely weakly compact in F, and consequently in E.

- Therefore

$$K_n = (n \cdot \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E$$

is absolutely weakly compact for every $n \in \mathbb{N}$,

- Therefore

$$K_n = (n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E$$

is absolutely weakly compact for every $n \in \mathbb{N}$, and by using an open cover argument, we obtain that $K = \bigcup_{n\in\mathbb{N}} K_n$ is also absolutely weakly compact.

- Therefore

$$K_n = (n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E$$

is absolutely weakly compact for every $n \in \mathbb{N}$, and by using an open cover argument, we obtain that $K = \bigcup_{n \in \mathbb{N}} K_n$ is also absolutely weakly compact.

- By the assumption, there exists an absolutely weakly null sequence $(y_n)_n$ in E such that K is contained in the closed convex hull of $\{y_n : n \in \mathbb{N}\}$, that is

$$\bigcup_{n=1}^{\infty} (n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E \subset \overline{co}(\{y_n : n \in \mathbb{N}\}),$$

which contradicts Lemma 1, since absolutely weakly null sequences are weakly null.

- Therefore

$$K_n = (n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E$$

is absolutely weakly compact for every $n \in \mathbb{N}$, and by using an open cover argument, we obtain that $K = \bigcup_{n \in \mathbb{N}} K_n$ is also absolutely weakly compact.

- By the assumption, there exists an absolutely weakly null sequence $(y_n)_n$ in E such that K is contained in the closed convex hull of $\{y_n : n \in \mathbb{N}\}$, that is

$$\bigcup_{n=1}^{\infty} (n \cdot \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})) \cap \frac{1}{n} B_E \subset \overline{co}(\{y_n : n \in \mathbb{N}\}),$$

which contradicts Lemma 1, since absolutely weakly null sequences are weakly null.

- Therefore, E has the positive Schur property, and we are done. □

Application

Application

- Recall that a Banach lattice E has the dual positive Schur property if every positive weak* null sequence in E^* is norm null, that is
$$0 \leq x_n^* \overset{\omega^*}{\to} 0 \text{ in } E^* \Rightarrow \|x_n^*\| \to 0.$$

Application

- Recall that a Banach lattice E has the dual positive Schur property if every positive weak* null sequence in E^* is norm null, that is
$$0 \leq x_n^* \overset{\omega^*}{\to} 0 \text{ in } E^* \Rightarrow \|x_n^*\| \to 0.$$
- Recall that a Banach lattice E has the positive Grothendieck property if every positive weak* null sequence in E^* is weakly null, that is
$$0 \leq x_n^* \overset{\omega^*}{\to} 0 \text{ in } E^* \Rightarrow x_n \overset{\omega}{\to} 0.$$

Application

- Recall that a Banach lattice E has the dual positive Schur property if every positive weak* null sequence in E^* is norm null, that is
$$0 \leq x_n^* \overset{\omega^*}{\to} 0 \text{ in } E^* \Rightarrow \|x_n^*\| \to 0.$$
- Recall that a Banach lattice E has the positive Grothendieck property if every positive weak* null sequence in E^* is weakly null, that is
$$0 \leq x_n^* \overset{\omega^*}{\to} 0 \text{ in } E^* \Rightarrow x_n \overset{\omega}{\to} 0.$$

Proposition (Wnuk)

A Banach lattice E has the dual positive Schur property if and only if E has both the positive Grothendieck property and E^* has the positive Schur property.

W. Wnuk, *On the dual positive Schur property in Banach lattices*, Positivity 17 (2013), 759-773.

Corollary

The following are equivalent for a Banach lattice E:
(a) E has the dual positive Schur property.
(b) Every absolutely weak* null sequence in E^* is norm null.
(c) E has the positive Grothendieck property and every sequentially absolutely weak*-compact subset of E^* is contained in the closed convex hull of an absolutely weak* null sequence.

Thank you!