Weak essential norms of pointwise multipliers between distinct Banach function spaces

Positivity XI

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Essential norms of the multiplication operators: The non-algebraic case
preprint available on arXiv.org (2023)

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- Sinnamon's Down spaces $X^\downarrow$, and so on...

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- but, in general, not Musielak–Orlicz spaces, Cesáro spaces, Tandori spaces, and so on...

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- Computing $M(X, Y)$ and describing $M(X, Y)_o$ are challenging tasks in themselves

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- Since $\mathscr{K}(X, Y) \subset \mathscr{W}(X, Y)$, so

$$\|T\colon X \to Y\|_{w} \leqslant \|T\colon X \to Y\|_{ess}. \tag{6}$$

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Useful note

Positive Schur = 1-DH!

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- Positive Schur spaces are somehow "close" to L_1.

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$$\mathcal{U}_n(\lambda) := \bigcup_{N \geqslant n} \Omega_N \cup \{\omega \in \Omega \colon |\lambda(\omega)| > n\} \tag{9}$$

and $\{\Omega_N\}_{N=1}^{\infty}$ is a decomposition of Ω into a pairwise disjoint family of sets of finite measure.

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$$\operatorname{dist}(f, X_o) = \operatorname{dist}(f^*, X_o). \tag{11}$$

Corollary [J. Tomaszewski and K, 2023]

Let X and Y be two rearrangement invariant function spaces with the Fatou property both defined on the same measure space.

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Let X and Y be two Banach function spaces with the Fatou property both defined on the same measure space. Suppose that X is reflexive and Y has the positive Schur property. Then

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Let X and Y be two Banach function spaces with the Fatou property both defined on the same measure space. Suppose that X is reflexive and Y has the positive Schur property. Then $M(X, Y)$ has an order continuous norm.

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Let $1 < p < q < \infty$. Then

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$$\mathscr{L}(L_p(0,1), L_{p,q}(0,1)) = \mathscr{W}(L_p(0,1), L_{p,q}(0,1)). \tag{15}$$

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Thank you for your attention!