The norm of the Cesàro operator minus the identity acting on decreasing sequences.

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Índex

Preliminaries

Recall that, for $\{x_n\}_n \subset \mathbb{R}$, the Cesàro operator C is defined as

$$Cx(n) = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Similarly, the transpose Copson operator C^* is given by

$$C^*x(n) = \sum_{k=n}^{\infty} \frac{x_k}{k}.$$

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G. Bennett [MAMS 1996] asked the question of finding the exact norm of $C - I$ in ℓ^p, which was shown by Brown, Halmos and A.L. Shields [ASM, 1965] to be 1, in ℓ^2.

G. J. O. Jameson [MIA, 2021] conjectured that, for $1 < p \leq 2$, $\|C - I\|_p = 1/(p-1)$ and he obtained the result for $p = 4/3$.

In [MIA, 2022] G. Sinnamon proved the conjecture and completely answered Bennet's question for the full range $1 < p \leq \infty$ (with a different value of the norm for $2 < p \leq \infty$).

Preliminaries

Similarly to what happens in the case of the Hardy operator defined for functions in $L^p(\mathbb{R}^+)$, if we restrict the action of $C - I$ to the cone of monotone decreasing and positive sequences in ℓ^p, denoted as ℓ^p_{dec}, the question of determining the exact norm of this restriction is very deeply related to the following theorem [G.Bennett, MAMS, 1996]:

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Theorem

If $2 \leq p < \infty$ and $x = \{x_n\}_n$ is a positive sequence, then

$$\|Cx\|_{\ell^p} \leq \zeta(p)^{1/p} \|C^* x\|_{\ell^p},$$

where

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad 1 < p < \infty,$$

is the Riemann zeta function and $\zeta(p)^{1/p}$ is the best possible constant.

Preliminaries

The case $1 < p < 2$ in the previous theorem is still an open question. The best estimates known are due to V. I. Kolyada [UMJ, 2019], who has recently obtained the following partial result:

Theorem

For $1 < p < 2$ and $x = \{x_n\}_n$ any positive sequence, the best constant A_p in the inequality

$$\|Cx\|_{\ell^p} \leq A_p \|C^*x\|_{\ell^p},$$

satisfies

$$\frac{1}{p-1} \leq A_p \leq \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}}.$$

Preliminaries

In this talk, we will prove optimal estimates for the norm of the Cesàro operator minus the identity, as well as C minus the shift operator $\sigma x(n) = x_{n+1}$, both acting on positive and decreasing sequences. This duality (the identity versus the shift) is a new and intrinsic fact in the discrete setting, which does not appear in the continuous case.

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In this talk, we will prove optimal estimates for the norm of the Cesàro operator minus the identity, as well as C minus the shift operator $\sigma x(n) = x_{n+1}$, both acting on positive and decreasing sequences. This duality (the identity versus the shift) is a new and intrinsic fact in the discrete setting, which does not appear in the continuous case. If we recall the definitions of the Hardy operator H and its conjugate H^*:

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \;\; H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt,$$

then

$$HH^*f = H^*Hf = Hf + H^*f,$$

while

$$CC^*(x) = C^*C(x) = C(x) + \sigma C^*(x).$$

Preliminaries

The basic remark in this context is that any decreasing and positive sequence is the image by the Copson operator of a positive sequence, and the estimate for $C - \sigma$ on decreasing sequences is equivalent to

$$\|Cx - \sigma x\|_{\ell^p} \leq A_p \|x\|_{\ell^p_{dec}} \Leftrightarrow \|Cx\|_{\ell^p} \leq A_p \|C^* x\|_{\ell^p}. \tag{1}$$

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$$\|Cx - \sigma x\|_{\ell^p} \leq A_p \|x\|_{\ell^p_{dec}} \Leftrightarrow \|Cx\|_{\ell^p} \leq A_p \|C^* x\|_{\ell^p}. \quad (1)$$

In what follows, for an operator T we will use the notation

$$\|T\|_{\ell^p_{\mathrm{dec}}} = \sup\{\|Tx\|_{\ell^p} : \ x \text{ is a decreasing sequence, } \|x\|_{\ell^p} = 1\}.$$

With this notation, and using the equivalence in (1), G. Bennett's theorem, and V. Kolyada's estimates, read as follows:

Corollary

If $2 \leq p < \infty$, *then*

$$\|C - \sigma\|_{\ell^p_{\mathrm{dec}}} = \zeta(p)^{1/p}.$$

If $1 < p < 2$, *then*

$$\frac{1}{p-1} \leq \|C - \sigma\|_{\ell^p_{\mathrm{dec}}} \leq \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}}.$$

Main result: unweighted case

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Theorem

Let $1 < p \leq 2$, *then*

$$\|C - I\|_{\ell^p_{\mathrm{dec}}} = \frac{1}{p-1},$$

and in the case $2 \leq p < \infty$,

$$\|C - I\|_{\ell^p_{\mathrm{dec}}} = \frac{1}{(p-1)^{1/p}}.$$

For $p = \infty$, *we have*

$$\|C - \sigma\|_{\ell^\infty_{\mathrm{dec}}} = \|C - I\|_{\ell^\infty_{\mathrm{dec}}} = 1.$$

Main result: weighted case

The result is consequence of Sinnamon's result, in the case $1 < p \leq 2$, and some estimates concerning C and C^* due to Bennett, in the case $p \geq 2$. But, in this case, the result can also be obtained as a particular case of the weighted sharp estimates.

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We recall that the weighted class B_p of Ariño-Muckenhoupt is defined as the collection of all positive and measurable functions w for which there exists a positive constant $C > 0$ such that

$$r^p \int_r^\infty \frac{w(x)}{x^p}\, dx \leq C \int_0^r w(x)\, dx.$$

For a weight $w \in B_p$, we denote by

$$\|w\|_{B_p} = \sup_{r>0} \frac{r^p \displaystyle\int_r^\infty \frac{w(x)}{x^p}\, dx}{\displaystyle\int_0^r w(x)\, dx}.$$

Main result: weighted case

This class of weights characterizes the weighted Hardy's inequalities, restricted to $L^p_{\mathrm{dec}}(w)$, the cone of positive and monotone decreasing functions in $L^p(w)$. Moreover, in some cases it also gives the precise estimate for the operator $S - I$:

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Theorem (B., Soria. JFA (2011))

If $p \geq 2$ and $w \in B_p$ satisfies that

$$r^{p-1} \int_r^\infty \frac{w(x)}{x^p}\, dx, \quad \textit{is a decreasing function of } r > 0,$$

then the following best estimate holds,

$$\|S - I\|_{L^p_{\mathrm{dec}}(w)} = (\|w\|_{B_p})^{1/p}.$$

Our goal is to find the analogous result, in the discrete setting. We will see that two different conditions of the weighted class come up very naturally.

Main result: weighted case

Given $w = \{w_n\}_n$ a sequence of positive numbers, we denote also by B_p the discrete class of weights that characterizes the boundedness of the Cesàro operator acting on decreasing and positive sequences. That is, the class of weights (see G. Bennett and K.G.-Grosse-Erdmann)

$$\|w\|_{B_p} = \sup_{n\in\mathbb{N}} \frac{\displaystyle\sum_{j=n}^{\infty} \frac{w_j}{j^p}}{\displaystyle\frac{1}{n^p}\sum_{j=1}^{n} w_j} < +\infty.$$

Clearly, this is equivalent to the class of sequences for which

$$\|w\|_{\widetilde{B_p}} = \sup_{n\in\mathbb{N}} \frac{\displaystyle\sum_{j=n+1}^{\infty} \frac{w_j}{j^p}}{\displaystyle\frac{1}{n^p}\sum_{j=1}^{n} w_j} < +\infty,$$

Main result: weighted case

In fact,

$$\|w\|_{\widetilde{B_p}} \leq \|w\|_{B_p} \leq 1 + \|w\|_{\widetilde{B_p}}.$$

Theorem

If $-1 < \alpha < p-1$, then the power weights $w_\alpha = \{w_{k,\alpha}\}_k = \{k^\alpha\}_k \in B_p = \widetilde{B_p}$ and,

(a) *for $0 \leq \alpha < p-1$, $\|w_\alpha\|_{\widetilde{B_p}} = \dfrac{\alpha+1}{p-\alpha-1}$;*

(b) *for $-1 < \alpha \leq 0$, $\|w_\alpha\|_{B_p} = \zeta(p-\alpha)$.*

In particular,

$$\|1\|_{B_p} = \zeta(p), \text{ and } \|1\|_{\widetilde{B_p}} = \frac{1}{p-1}.$$

Main result: weighted case

Theorem

Let $p \geq 2$*, and* $w = \{w_n\}_n \in B_p$ *be a weight satisfying the monotonicity condition*

$$\frac{1}{n}\sum_{k=1}^{n} w_k \quad \text{is a decreasing sequence in } n. \tag{2}$$

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Then

$$\|C - \sigma\|_{\ell^p_{\mathrm{dec}}(w)} = \left(\|w\|_{B_p}\right)^{1/p}$$

and

$$\|C - I\|_{\ell^p_{\mathrm{dec}}(w)} = \left(\|w\|_{\widetilde{B_p}}\right)^{1/p}.$$

Condition (2) cannot be dropped: if $x = \{3, 2, 1, 0, 0, \cdots\}$,

$$\|(C-\sigma)\|^2_{\ell^2(\{\sqrt{n}\}_n)} \geq \frac{\|(C-\sigma)x\|^2_{\ell^2(\{\sqrt{n}\}_n)}}{\|x\|^2_{\ell^2(\{\sqrt{n}\}_n)}} = 3.02031 > \|\{\sqrt{n}\}_n\|_{B_2} = 3.$$

Main result: weighted case

The previous theorem is false, in general, for $C - I$ on the range $1 < p < 2$, shows for the constant weight $w = 1$:

$$\|C - I\|_{\ell^p_{\mathrm{dec}}} = \frac{1}{p-1} \neq \left(\|1\|_{\widetilde{B_p}}\right)^{1/p} = \left(\frac{1}{p-1}\right)^{1/p}.$$

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For $C - \sigma$ and $1 < p < 1.59025...$ (the value where $\frac{1}{p-1} = \zeta(p)^{1/p}$), using the previous mentioned corollary from Kolyada's estimates, we have a similar counterexample:

$$\|C - \sigma\|_{\ell^p_{\mathrm{dec}}} \geq \frac{1}{p-1} > \left(\|1\|_{B_p}\right)^{1/p} = \zeta(p)^{1/p}.$$

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However, both estimates, for $C - I$ and $C - \sigma$, remain true if $p = 1$,without any other hypothesis on the weight

$$\|C - I\|_{\ell^1_{\rm dec}(w)} = \|w\|_{\widetilde{B_1}} \text{ and } \|C - \sigma\|_{\ell^1_{\rm dec}(w)} = \|w\|_{B_1}.$$

Related results. Open questions

- Find the best estimate in the range $1 < p < 2$ such that, for any decreasing sequence x,

$$\|(C-\sigma)x\|_{\ell^p} \le A_p \|x\|_{\ell^p_{\mathrm{dec}}}.$$

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- Restricted to positive sequences, the following analogous result about optimal constants holds (B., A. Ben Said, J. Soria)

$$\|C - I\|_{\ell^p_{\text{pos}}} = 1, \text{ for } p \geq 2,$$

$$\|C - I\|_{\ell^p_{\text{pos}}} = \frac{1}{p-1}, \text{ for } 1 < p \leq 2.$$

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