A Stone-Weierstrass type theorem for truncated vector lattices of functions

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    - A **Banach lattice algebra of functions** on X is a Banach lattice of functions on X which is simultanouesly a Banach algebra of functions on X.

Let X be a topological space. A continuous real-valued function f on a topological space X is said to be **vanishing at infinity** if, for every $\varepsilon > 0$, the set

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Example

For each real number a, let f_a denote the real-valued function on the real interval $X = [0;1]$ defined by

$$f_a(x) = ax \quad (x \in X)$$

Consider

$$B(X) = \{f_a : a \in \mathbb{R}\}.$$

Problem

What is the missing condition for a Banach lattice of functions on X to be a Banach algebra of functions on X?

Definition

A non-empty set $S(X)$ of real-valued functions on X is said to have the **Stone prpoperty** or to be **truncated** if

$$1 \wedge f \in L \quad \text{for all } f \in L.$$

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Theorem

A set $B(X)$ of real-valued functions on X is a Banach algebra of functions on X if and only if $B(X)$ a truncated Banach lattice of functions on X.

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- **vanish nowhere** if there is no $x \in X$ such that $f(x) = 0$ for all $f \in S$.

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Lemma

Let X be a locally compact Hausdorff space and L be a truncated vector sublattice of $C_0(X)$. If L separates the points of X and vanishes nowhere on X then L is uniformly dense in $C_0(X)$.

Definition

A non empty set S of $B(X)$ is **algebraically dense** in $B(X)$ if for every algebra homomorphisms δ_1 and δ_2 on $B(X)$, the equality $\delta_1 = \delta_2$ holds whenever $\delta_1 f = \delta_2 f$ for all $f \in S$.

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Theorem

Let $B(X)$ be a Banach algebra on a non-empty set X. If L is either a subalgebra or a truncated vector sublattice of $B(X)$ then L is algebraically dense in $B(X)$ if and only if L is uniformly dense in $B(X)$.

Definition

A Banach algebra $B(X)$ on a non empty set X is said to be $B(X)$**-realcompact** if every nonzero (real-valued) algebra homomorphism on $B(X)$ is an evaluation δ_x at some $x \in X$, where

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Corollary

Let $B(X)$ be a Banach algebra on a non empty set X which is $B(X)$-realcompact and E be either a subalgebra or a truncated vector sublattice of $B(X)$. If E separates the points of X and vanishes nowhere on X then E is uniformly dense in $B(X)$.

Corollary

Let $L(X)$ *be a truncated vector sublattice of* $\ell^{\infty}(X)$*. Then there exists a locally compact Hausdorff space* Y *and a linear map* $T : L(X) \to C_0(Y)$ *such that*

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Corollary

Let $L(X)$ be a truncated vector sublattice of $\ell^{\infty}(X)$. Then there exists a locally compact Hausdorff space Y and a linear map $T : L(X) \longrightarrow C_0(Y)$ such that

1. *T is an isometry,*
2. *T is a lattice homomorphism with uniformly dense range (in $C_0(Y)$), and*
3. *T preserves truncation, i.e.,*

$$T(1 \wedge f) = 1 \wedge Tf \quad (f \in L(X)).$$

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Corollary

For any truncated Banach lattice $B(X)$ there exists a locally compact Hausdorff space Y such that $B(X) = C_0(Y)$.

Representation of truncated vector sublattices of AM spaces.

A norm on a real vector lattice L is called an M-norm if $|f| \leq |g|$ implies $||f|| \leq ||g||$ in L and $||f \vee g|| = \max\{||f||, ||g||\}$ for all f, g positive in L. If in addition the M-norm is complete, we call L an Abstract M-space (briefly, an AM-space). An element $e > 0$ in the AM-space L is called a (strong) unit if the set $\{f \in L : -e \leq f \leq e\}$ coincides with the closed ball of L.

Theorem (The Kakutani-Krein Representation)

For any AM-space with unit $e > 0$, there exists a compact Hausdorff space X such that L and $C(X)$ are lattice isometric.

Definition

A vector sublattice of *AM*-space E with unit $e > 0$ is **truncated** if $e \wedge f \in L$ for all $f \in L$.

Corollary

Let E be an AM-space with unit $e > 0$. Then any closed truncated vector sublattice of E is lattice isometric to $C_0(Y)$ for some locally compact Hausdorff space Y.

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