ON LIMITED AND ALMOST LIMITED OPERATORS BETWEEN BANACH LATTICES

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Recall that an operator $T:X\to F$ is called semi-compact if it takes bounded sets of X onto almost order bounded subsets of F. Compact operators are semi-compact. Alternatively, T is semi-compact if for each $\varepsilon > 0$, there exists $u \in F^+$ such that $T(B_X) \subseteq [-u,u] + \varepsilon B_F$.

An operator $T:E\to X$ is called M-weakly compact if $||Tx_n||\to 0$ for each disjoint bounded sequence (x_n) in E.

An operator $T: X\to E$ is called L-weakly compact if $||y_n|| \to 0$ for each disjoint sequence (y_n) in the solid hull of $T(B_X)$.

A Banach lattice E has property (d) if $|f_n|\to 0$ for every disjoint weak*null sequence (f_n) in E*. If E is a σ-Dedekind complete Banach lattice, then E has (d). if E* has weak* sequentially continuous lattice operations then E has (d). $L^p[0,1]$, $1\leq p\leq \infty$ has (d) whereas C[0,1] does not have (d) see Example 2.2:3 in [3].

T: X→ F is called almost L-weakly compact (aLwc) if it maps relatively weakly compact subsets of X onto L-weakly compact subsets in F. Or $f_n(Tx_n)\to 0$ for each disjoint bounded (f_n) in F* and weakly null (x_n) in B_E and T(X)⊆ F^a.

T: E→ X is called almost M-weakly compact (aMwc) if for each disjoint sequence (x_n) in B_E and weakly null (f_n) in X*, $f_n(Tx_n)\to 0$.

aMwc operators contain M-weakly compact operators. *Id* of $\boldsymbol{l}^\infty$ is aMwc but it is not M-weakly compact.

Order intervals of E are limited if and only if lattice operations in E* are weak*sequentially continuous.

A norm bounded subset A of X is limited iff $f_n(a_n) \to 0$ for all weak* null (f_n) in X* and all (a_n) in A.
Compact subsets in X are limited. If each limited subset of X is relatively compact then X is said to be a Gelfand-Phillips (GP-space, for short). Reflexive spaces, separable spaces, σ-Dedekind complete Banach lattices with order continuous norms are GP-spaces. l^1 and c_0 are GP-spaces but l^∞ is not. Order intervals of a Banach lattice are not necessarly limited sets. For example the interval [-1,1] in c is not limited

If F is a KB-space, then for every Banach lattice E where E* has PSP, each regular operator is M and L- weakly compact.(Theorem 3.8)

If E* is a KB-space, then for every Banach lattice F with PSP every regular operator $T:E \to F$ is M and L-weakly compact.(Theorem 3.8)

Assertion 1) If F is discrete with order continuous norm, then every regular Mwc T:E→X is compact.(Theorem 3.4)

If E* is discrete with order continuous norm, then every regular Lwc operator T:E→F is compact (Theorem 4.2)

If Y contains c_0, then every Mwc T:E→Y is compact iff $(E^*)^a$ is discrete.

An operator T: X→Y is called limited if $T(B_X)$ is a limited subset of Y. T: X→E is called almost limited if $T(B_X)$ is almost limited in E.The embedding of c_0 into l^∞ is limited, but not compact. The closed unit ball of *l*∞ is almost limited.Phillip's lemma (Theorem54.67 [2] says closed unit ball of c_0 is limited in l∞

Let T:X→F and suppose F has order continuous norm. If T is limited then it is L-weakly compact and therefore is weakly compact and semi-compact.

The adjoint T^* is almost Dunford-Pettis and order weakly compact. If F^* has weak* continuous lattice operations, then limited operators is an order ideal in L-weakly compact operators.

THE EFFECT OF WEAK* CONTINUITY OF LATTICE OPERATIONS IN THE DUAL

A operator $T:X\to F$ is called limitedly *L*-weakly compact(*l*-Lwc) if T maps limited subsets of X onto L-weakly compact subsets of F.

l-Lwc operators were defined and studied in [3]. An operator T is *l*-Lwc if and only if $T^* f_n\to 0$ in weak* topology of X^* for each bounded disjoint (f_n) by Lemma 2.3.1. in [3].

Let $T:X\to F$ be a semi-compact operator:

- *If F has limited order intervals, then T is limited.*
- *If F has almost limited intervals, then T is almost limited.*
- *If F has compact order intervals, then T is compact.*
- *If F has order continuous norm, then T is weakly compact*

It follows that L-weakly compact, order bounded M-weakly compact, and operators dominated by semi-compact operators taking values in a Riesz space with limited order intervals are limited operators as each of these operators are semi-compact. See Prop.2.1. in [5].

A semi-compact operator need not be compact, weakly compact, L or M-weakly compact. The identity of l^∞ is semi-compact but it does not have any of the compactness properties mentioned.

Let T: E→ F be a semi-compact operator. If F has limited order intervals and order continuous norm, then T is compact.

Let E have limited order intervals and F have order continuous norm and let T be a regular semi-compact operator E→ F, then T is AM-compact.

Let T: E → X be a weakly compact operator. If E has limited order intervals then T is AM-compact

Let us note that not every weakly compact operator is AM-compact. The identity of $L^2[0,1]$ is weakly compact but it is not AM-compact.

Suppose E has limited order intervals. Let T: E → X be order weakly compact, then T is AM-compact

Suppose E has order continuous norm and E has limited order intervals and F has PSP, then every order bounded weakly compact operator T: E→F is limited.*

Proof. Since F has PSP, F has order continuous norm. As E* and F has order continuous norms an order bounded T:E→F is compact if and only if T is semi-compact and AM compact by Theorem 125.5 in [22]. Let T: E→F be a weakly compact operator. Then T is AM compact by Proposition 2.2. above. On the other hand T is M-weakly compact by Theorem 3.3 in [11]. As order bounded M-weakly compact operators are semi-compact, T is semi-compact and therefore compact.■

Suppose E has order continuous norm and F* has weak* continuous lattice operations, then each positive aDP operator is limited*.

Suppose E has weak* sequentially continuous lattice operations, then each Mwc operator T: E→F is limited.*

Proof. Let (f_n) be a weak*null sequence in F*. We need to show $\|T^*(f_n)\|\to 0$. For this it suffices to show $|T^*(f_n)|\to 0$ in $\sigma(E^*,E)$ and $T^*(f_n)(x_n)\to 0$ for each norm bounded disjoint sequence (x_n) in E_+. Since T is bounded $(T^*f_n)\to 0$ in $\sigma(E^*,E)$. Continuity of lattice operations in E* gives us $|T^*f_n|\to 0$ in $\sigma(E^*,E)$.
$|T^*(f_n)(x_n)| = |f_n(Tx_n)|\leq\|f_n\|\,\|Tx_n\|$
Since (f_n) is norm bounded and T is Mwc, The claim follows.■

If every positive limited operator T: E→F is Mwc, then E is a KB space.*

Proof. Assume E* does not have order continuous norm. Then there exists an order bounded disjoint sequence in E* such that $(f_n)\subseteq[o,f]$ which does not converge to zero in norm. Choose $y\varepsilon F_+$ such that $\|y\| =1$ and a functional $g\varepsilon F^*$ with $\|g\| = 1$ and $g(y) = \|g\| =1$. Define T: E→F by $T(x) = g(x)y$ for xεE. As T is compact, it is limited. We claim T is not Mwc. For this it is enough to show T* is not Lwc. Let us observe that $T^*(h) = h(y)f$. Taking h=g, we see that $T^*g =f$. So that the sequence (f_n) is a sequence in the solid hull of $T^*(B_F)$. This shows that T*is not Lwc and therefore T is not Mwc.■

If Dunford-Pettis operators from E to F are limited, then one of E^* has order continuous norm, or order bounded subsets of F are limited holds.

If every weakly compact T: E→X is limited, then one of the following holds:

- *X has the Dunford-Pettis property,*
- *E^* has order continuous norm,*

If each semi-compact operator T :X→F is aLwc for every X then F has order continuous norm.

Each limited operator T:E→F is aLwc if and only if F has order continuous norm.

1) *If each limited operator T:E→F is aMwc then, E^* has order continuous norm.*

2) *If E^* has order continuous norm, then each regular T:E→F limited operator is aMwc.*

If E has order continuous norm and limited order intervals, then T^2 is compact for each positive semi-compact operator on E.

Suppose E has limited order intervals and T: E→E be semi-compact and weakly compact. Then T^2 compact.

an operator T:E→F is called limitedy M-weakly compact if, each bounded disjoint sequence (x_n) in E, (Tx_n) is weakly null.

Suppose F has Schur property, then l -Mwc(E,F) ⊆ W_o(E,F).

Proof. Let T ε *l*-Mwc. We need to show $||Tx_n|| \to 0$ for each order bounded disjoint sequence (x_n) in E. (x_n) is weakly null as it is order bounded and disjoint. Thus, (x_n) is bounded, as T is *l*-Mwc, (Tx_n) is weakly null and therefore norm null by the Schur property of F.■

ALMOST LIMITED OPERATORS

Recall that E has positive disjoint Schur property(PDSP) if every positive disjoint weak* null sequence in the dual is norm null. It follows that whenever E has PDSP then every bounded operator into E is almost limited. Also notice that since each L-weakly compact set in a Banach lattice is almost limited by Theorem 2.6. in [10], each L-weakly compact operator $T:X \rightarrow E$ is almost limited.

Recall that a Banach lattice is said to have weak Dunford-Pettis* Property (wDP*P, for short) if every relatively weakly compact set is almost limited. if E has wDP*property then each weakly compact operator $T:X \rightarrow E$ is almost limited. In particular, every L and M-weakly compact operator is almost limited.

The identity of l^{∞} is almost limited but it is not limited as the closed unit ball of l^{∞} is not a limited set.

If F has dual disjoint Schur property then, each bounded operator T:E→F is almost limited.

If F has (d) then, each regular Mwc T:E→F is almost limited.

Let (f_n) be a disjoint weak*null sequence in E*.By the (d) property of F,$(|f_n|)\to 0$ is also weak*null in F*.As $|T^*f_n|\leq T^*|f_n|$ the sequence $(|T^*f_n|)$ is also weak*null in E*.Thus to show $||T^*f_n||\to 0$, it suffices to show $T^*f_n(x_n)\to 0$ for each disjoint bounded sequence (x_n) in E_+ Consider

$|T^*f_n(x_n)| = |f_n(Tx_n)| \leq ||f_n|| \, ||Tx_n||$

Since $||f_n|| \leq N$ for some N and all n, and since $||Tx_n||\to 0$ as T is Mwc,it follows that $|T^*f_n(x_n)|\to 0$ and therefore $||T^*f_n||\to 0$ and T is almost limited.■

Corollary If E*has positive Schur property, then for Banach lattice F with (d) each T:E→ F weakly compact is almost limited. ■

Corollary Suppose E*has the PSP and F is a KB space. Then every regular T:E→F is almost limited.

Proof. As they are L and M weakly compact operators.■

For a Banach lattice F, the following are equivalent;
•*F has order continuous norm,*
• *For every X, each almost limited operator T:X→F is Lwc,*
•*For every X , each limited operator T:X→F is Lwc,*
•*For every E, each positive rank one operator is Lwc*.

iv) implies i).For each y$\varepsilon F_{+,}$ there exits a positive operator T E→F such that T(E) = span(y), then by [3, Theorem 5.66] ,y$\varepsilon F^{a,}$ and as yεF_{+} is arbitrary, this implies that F = F^{a} and hence F has order continuous norm.■

Let F has positive Grothendieck property and has (d), then every aMwc operator is almost limited.

Suppose E has (d) and let T:E→F. If T is both l-Lwc and Mwc, then T is almost limited.

Proof. We need to show $||T^*f_n||\to 0$ for each weak*null disjoint sequence (f_n) in F*. For this it suffices to show $|T^*f_n| \to 0$ weak* and $(T^*f_n)(x_n)\to 0$ for each disjoint bounded sequence (x_n) in E_+ .Since T is *l*-Lwc (T^*f_n) is weak* null in E*. Property (d) ensures that $|T^*f_n|\to 0$ in weak* topology. Since (f_n) is weak*null, it is bounded and it follows from
$|T^*(f_n(x_n)|=|f_n(Tx_n)|\leq ||f_n||||Tx_n||$
AsT is Mwc $||Tx_n|| \to 0$ and $T^*(f_n(x_n)) \to 0$

Suppose each positive weak Dunford-Pettis operator from E to F is Mwc and F is σ-Dedekind complete then one of E has positive Schur property or F has order continuous norm holds.

Let E, F be Banach lattices where E has almost-DP property. F is σ-Dedekind complete and E* has order continuous norm, then each regular operator T: E→ F is almost limited.*

Corollary Suppose E is a Banach lattice such that E* has the PSP, then for each KB-space F and regular operator T:E→F, T is almost limited.

Corollary. Suppose F is a KB-space. Then for every Banach lattice E with E* has PSP, every regular operator is almost limited.

Corollary. Suppose E* is a KB-space, then for every Banach lattice with PSP, each regular operator is almost limited.

Every positive aDPO is almost limited iff F has DDSP or E* has order continuous norm.

Suppose F has (d) and DPSP (or equivalently F has DDSP) or E is KB, then each positive aLwc operator is almost limited*.

Not every almost limited operator is M-weakly compact.The identity on l^∞ is almost limited but it is neither M nor L-weakly compact

Suppose E has order continuous norm and F has (d). If every positive almost limited operator T: E→ F is Mwc, then E and F have order continuous norms*.

Almost limited operators are contained in aLwc (X,E) operators iff E has order continuous norm.

Each almost limited positive operator T: E→ F is aMwc iff E has order continuous norm.*

Suppose E has order continuous norm and each positive weakly compact operator T: X→ E is almost limited. Then one of E is a KB-space or X has the Dunford-Pettis Property holds.

Let E and F be σ-Dedekind complete Banach lattices. If one of E or F has order continuous norm, then each almost limited T:E→F is order weakly compact.

Suppose E and F have order continuous norms and E has the property that each bounded positive disjoint sequence in E is order bounded and E has limited order intervals. Then each regular operator T:E→F is positively limited.*

Let E be an AL-space and F be a Banach lattice with order continuous norm, then each almost
Limited operatorT :E →F has an almost limited modulus.|T| .

Proof. Since F has order continuous norm, almost limited T is L-weakly compact. Hence T has an L-weakly compact modulus. |T| by Theorem2.4. in[11]. Thus |T| is almost limited by Theorem 2.6 in[8].

It is well known that adjoints of almost limited operators are almost Dunford-Pettis and order weakly compact. On the other hand an almost limited operator itself need not be order weakly compact. For example the identity on l^∞ is almost limited but not order weakly compact. Duality of order weakly compact operators was studied in [6].

Let E and F be σ-Dedekind complete Banach lattices. If one of E or F has order continuous norm, then each almost limited T:E→F is order weakly compact.

Proof. Let T:E→F be almost limited. Then T* is order weakly compact. Then T is also order weakly compact by Theorem 2.8 in [6].■

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The following result was proved for order weakly compact operators T:E→F where E has the property that each bounded disjoint sequence (x_n) is order bounded.

Proposition 3.19. Suppose E* and F have order continuous norms and E has the property that each bounded positive disjoint sequence in E is order bounded and E has limited order intervals. Then each regular operator T:E→F is almost limited.

Proof. The operator T admits a factorization over a Banach lattice G with G and G* have order continuous norms, say T = SR where R and S are both positive. We claim R is positively limited. Let (z_n^*) be a positive weak*null sequence in E* and (x_n) be a positive disjoint bounded sequence in E. By the assumption there exists x with $0 \le x_n \le x$ for each n. As R is positive $0 \le Rx_n \le Rx$. Thus $0 \le R^*(z_n^*)(x_n) \le (z_n^*)(R x_n) \le (z_n^*)(x)$ which shows $R^*(z_n^*)(x_n) \to 0$. By the Lemma we conclude that R is positively limited. Let (x_n^{**}) be a positive weak*null sequence in E^{**} (y_n^*) be a positive disjoint bounded sequence in F*. Since F hs order continuous norm, (y_n^*) is weak*-null and hence $R^*(y_n^*) \to 0$ in norm proving R is positive limited. It follows that T is almost limited as S is bounded.■