Weak compactness in variable exponent Lebesgue spaces

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Variable Lebesgue spaces

Let (Ω, Σ, μ) be a finite or σ-finite separable non-atomic measure space.

Definition

Given $p(\cdot) : \Omega \to [1, \infty)$, the variable exponent Lebesgue space (or Nakano space) $L^{p(\cdot)}(\Omega)$ is the Banach function space consisting of all $f \in L_0(\Omega, \mu)$ with

$$\rho\left(\frac{f}{r}\right) := \int_\Omega \left|\frac{f(t)}{r}\right|^{p(t)} d\mu(t) < \infty, \text{ for some } r > 0,$$

endowed with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf\left\{r > 0 \colon \rho\left(\frac{f}{r}\right) \leq 1\right\}.$$

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- $p^+ := \operatorname{ess\,sup}_{t\in\Omega} p(t)$.
- $p^- := \operatorname{ess\,inf}_{t\in\Omega} p(t)$.
- $R_{p(\cdot)} := \{q \in [1, \infty) \colon \forall \varepsilon > 0, \mu\left(p^{-1}(q - \varepsilon, q + \varepsilon)\right) > 0\}$.

Musielak-Orlicz spaces

Definition

A Musielak-Orlicz function is a measurable function $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ *such that* $\Phi(t,\cdot)$ *is an Orlicz function for every* $t \in \Omega$.

The Musielak-Orlicz space is the space of all $f \in L_0(\Omega)$ *with*

$$\int_\Omega \Phi\left(t, \left|\frac{f(t)}{r}\right|\right) dt < \infty, \text{ for some } r > 0,$$

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$$\|f\|_\Phi = \inf\left\{r > 0 \colon \int_\Omega \Phi\left(t, \left|\frac{f(t)}{r}\right|\right) dt \leq 1\right\}.$$

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- If $\Phi(t, x) = \varphi(x)$, then $L^\Phi(\Omega) = L^\varphi(\Omega)$ is an Orlicz space.
- If $\Phi(t, x) = x^{p(t)}$, then $L^\Phi(\Omega) = L^{p(\cdot)}(\Omega)$ is a variable exponent Lebesgue space.

Basic properties of $L^{p(\cdot)}(\Omega)$

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- Variable Lebesgue spaces are not rearrangement invariant (symmetric) Banach function spaces.
- ℓ_r is lattice embedded in $L^{p(\cdot)}(\Omega) \Leftrightarrow r \in R_{p(\cdot)}$.

Weak compactness in $L_1(\Omega)$

$S \subset L^{p(\cdot)}(\Omega)$ is **equi-integrable** if it BOTH is **uniformly integrable**, i.e.

$$\lim_{\mu(E)\to 0} \sup_{f\in S} \int_E |f(t)|^{p(t)} dt = 0$$

and **decays uniformly at infinity**, i.e. given $\varepsilon > 0$ there exists $A \subset \Omega$ with $\mu(A) < \infty$ such that

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Theorem (De La Vallée Poussin)

$S \subset L_1(\Omega)$ for a finite measure space Ω is equi-integrable $\Leftrightarrow$ S is norm bounded in some Orlicz space $L^\varphi(\Omega)$ for a N-function φ, i.e.

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Theorem (Dunford-Pettis)

$S \subset L_1(\Omega)$ is relatively weakly compact $\Leftrightarrow$ S is equi-integrable.

Equi-integrability in $L^{p(\cdot)}(\Omega)$ (for $\mu(\Omega) < \infty$)

For finite measures μ, $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable $\Leftrightarrow$

$$\lim_{x\to\infty} \sup_{f\in S} \int_{\{|f|>x\}} |f(t)|^{p(t)} d\mu = 0.$$

Theorem (De La Vallée-Poussin's type)

Let $L^{p(\cdot)}(\Omega)$ with $\mu(\Omega) < \infty$ and $p^+ < \infty$. $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable $\Leftrightarrow$ it is bounded and there exists an N-function φ such that

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There is not an analogous to Dunford-Pettis's theorem, since we can find non-equi-integrable weakly compact subsets in $L^{p(\cdot)}(\Omega)$ spaces. For example, $\ell_r \subsetneq L^{p(\cdot)}(\Omega)$ for $r \in R_{p(\cdot)}$ (and $r \neq 1$).

Weak compactness in Orlicz spaces (for $\mu(\Omega) < \infty$)

Theorem (Luxemburg)

$S \subset L^{\varphi}(\Omega)$ for an N-function φ satisfying the Δ_2-condition is relatively weakly compact $\Leftrightarrow$ for every $g \in L^{\varphi^*}(\Omega)$,

$$\lim_{\mu(E)\to 0} \sup_{f\in S} \int_E |f(t)\cdot g(t)|dt = 0$$

and, given $\varepsilon > 0$, there exists $A \subset \Omega$ with $\mu(A) < \infty$ such that

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Theorem (Ândo, Nowak)

$S \subset L^{\varphi}(\Omega)$ for an N-function φ satisfying the Δ_2-condition is relatively weakly compact $\Leftrightarrow$

$$\lim_{\lambda \to 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} \varphi\left(|\lambda \cdot f(t)|\right) dt = 0.$$

Weak compactness in $L^{p(\cdot)}(\Omega)$-spaces

Theorem (Luxemburg's type)

Let $L^{p(\cdot)}(\Omega)$ with $\mu(\Omega) < \infty$ and $p^+ < \infty$. $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact $\Leftrightarrow$ it is norm bounded and, for every $g \in L^{p^*(\cdot)}(\Omega)$,

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Denote $\Omega_1 := p^{-1}(\{1\})$.

Theorem (Andô's type)

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $\mu(\Omega_1) = 0$. $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact $\Leftrightarrow$

$$\lim_{\lambda\to 0} \sup_{f\in S} \frac{1}{\lambda} \int_\Omega |\lambda f(t)|^{p(t)} d\mu = 0.$$

Consequences

Proposition

Let $L^{p(\cdot)}(\Omega)$ with $\mu(\Omega) < \infty$ and $p^+ < \infty$. A sequence (f_n) in $L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega) \Leftrightarrow$

(i) $\lim_n \int_A (f_n - f) d\mu = 0$ *for each* $A \in \Sigma$, *and*

(ii) $\lim_{\mu(A)\to 0} \sup_n \int_A |(f_n - f)g| d\mu = 0$ *for each function* $g \in L^{p^*(\cdot)}(\Omega)$.

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Theorem

$L^{p(\cdot)}(\Omega)$ is weakly Banach-Saks $\Leftrightarrow p^+ < \infty$.

Corolary

Let ℓ_{p_n} be a Nakano sequence space with (p_n) bounded and $p_n \neq 1$. $S \subset \ell_{p_n}$ is relatively weakly compact $\Leftrightarrow$

$$\lim_{\lambda \to 0} \sup_{x=(x_n)\in S} \frac{1}{\lambda} \sum_{n=1}^{\infty} |\lambda x_n|^{p_n} = 0.$$

Corolary

Let ℓ_{p_n} with $p^+ < \infty$ and $p_n \neq 0$. A sequence $(x^k)_k$ in ℓ_{p_n} converges weakly to $y = (y_n) \in \ell_{p_n}$ $\Leftrightarrow$

(i) *For each coordinate n,* $\lim_{k\to\infty} x_n^k = y_n$, *and*

(ii) $\lim_{\lambda\to 0} \sup_{k\in\mathbb{N}} \frac{1}{\lambda} \sum_{n=1}^{\infty} |\lambda(x_n^k - y_n)|^{p_n} = 0.$

Corolary (Kaminska and Lee)

A Nakano sequence space ℓ_{p_n} is weakly Banach-Saks $\Leftrightarrow$ $p_n^+ < \infty$.

A criterion through Musielak-Orlicz spaces

Definition

A Musielak-Orlicz function $\Psi(t,x)$ *increases uniformly more rapidly than another function* $\Phi(t,x)$ *at infinity if,* $\forall \varepsilon > 0$ $\exists \delta > 0$ *and* $x_0 > 0$ *such that* $\forall x \geq x_0$ *and a.e-*$t \in \Omega$*,*

$$\varepsilon \Psi(t,x) \geq \frac{1}{\delta} \Phi(t, \delta x).$$

Theorem (Andô's type)

Let $L^{p(\cdot)}(\Omega)$ *with* $\mu(\Omega) < \infty$ *and* $p_+ < \infty$*.* $S \subset L^{p(\cdot)}(\Omega)$ *is relatively weakly compact* $\Leftrightarrow$ *there exists a Musielak-Orlicz function* $\Psi(t,x)$ *increasing uniformly more rapidly than* $\Phi(t,x) = x^{p(t)}$ *at infinity such that* S *is norm bounded in* $L^{\Psi}(\Omega)$*.*

Theorem (Nowak's type)

Let $L^{p(\cdot)}(\Omega)$ *be with* $\mu(\Omega) = \infty$ *and* $p_+ < \infty$*.* $S \subset L^{p(\cdot)}(\Omega)$ *is relatively weakly compact* $\Leftrightarrow$ *there exists a Musielak-Orlicz function* $\Psi(t,x)$ *increasing uniformly more rapidly than* $\Phi(t,x) = x^{p(t)}$ *for all* x *such that* S *is norm bounded in* $L^{\Psi}(\Omega)$*.*

Inclusions between variable Lebesgue spaces

Proposition

Let Ω be a non-atomic measure space and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. The inclusion $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ holds if and only if $p(t) \geq q(t)$ a.e-μ and there exists $\lambda > 1$ such that

$$\int_{\Omega_d} \lambda^{-r(t)} d\mu < \infty,$$

where $\Omega_d = \{t \in \Omega : p(t) > q(t)\}$ and $r(\cdot)$ is the defect exponent defined by

$$\frac{1}{r(\cdot)} := \frac{1}{q(\cdot)} - \frac{1}{p(\cdot)}.$$

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Note that, if $\mu(\Omega) < \infty$, then the condition $\int_{\Omega_d} \lambda^{-r(t)} d\mu < \infty$ is trivially satisfied, so the inclusion holds if and only if $p(\cdot) \geq q(\cdot)$. The atomic case will be considered later.

Weakly compact and L-weakly compact inclusions

Definition

- *An operator* $T : X \to Y$ *is weakly compact if the image of the unit ball* $T(B_X)$ *is a relatively weakly compact set in* Y*.*
- *An operator* $T : X \to E(\Omega)$ *is L-weakly compact if the image of the unit ball* $T(B_X)$ *is an equi-integrable set in* $E(\Omega)$*.*

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Proposition

The inclusion $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ *for bounded exponent functions* $p(\cdot) \geq q(\cdot)$ *and* $\mu(\Omega) < \infty$ *is L-weakly compact* $\Leftrightarrow$ *if there exists an N-function* φ *such that* $L^{p(\cdot)}(\Omega) \subset L^{\Phi}(\Omega)$, *where* Φ *is the Musielak-Orlicz function* $\Phi(t, x) := (\varphi(x))^{q(t)}$.

Proposition

The inclusion $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ *for bounded exponent functions* $p(\cdot) \geq q(\cdot)$ *and* $\mu(\Omega_1^{q(\cdot)}) = 0$ *is weakly compact* $\Leftrightarrow$

$$\lim_{\lambda \to 0} \sup_{\|f\|_{p(\cdot)} \leq 1} \frac{1}{\lambda} \int_{\Omega} |\lambda f(t)|^{q(t)} d\mu = 0.$$

There is more

We can say a bit more when Ω is non-atomic and $\mu(\Omega) = \infty$.

Theorem

Let the inclusion $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ hold for a non-atomic space Ω with $\mu(\Omega) = \infty$ and $p^+ < \infty$. Then, it is not L-weakly compact.

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Denote $\Omega^{p(\cdot)}_{1+\varepsilon} := \{x \in \Omega : p(x) < 1 + \varepsilon\}$.

Proposition

Let $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ for a non-atomic separable measure space. If $\mu(\Omega^{p(\cdot)}_{1+\varepsilon}) = \infty$ for every $\varepsilon > 0$, then the inclusion $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is not weakly compact.

THANK YOU VERY MUCH