Representing Noetherian vector lattices

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Noetherian vector lattice

Inspired by ring theory:

Definition

A vector lattice E is *Noetherian* if every increasing sequence of ideals stabilizes (i.e., is eventually constant).

Examples?

- If E is Archimedean and infinite-dimensional, then it contains a disjoint sequence (u_n). Then $(I_{u_1+\ldots+u_n})_{n=1}^{\infty}$ is a strictly increasing sequence of ideals.
- So if E is Archimedean, it is Noetherian if and only if it is finite-dimensional ($\cong \mathbb{R}^n$)

Next slide: towards finding non-Archimedean examples.

Let γ be an ordinal.

- Vector space: $\mathbb{R}^\gamma_{\mathrm{lex}} = \{f : \gamma \to \mathbb{R}\}$
- Ordered by cone C of functions f that are positive in the smallest element of its support.

Is $\mathbb{R}^\gamma_{\mathrm{lex}}$ Noetherian?

- For $\beta \in \gamma$, define $e_\beta(\alpha) := \delta_{\alpha\beta}$
- Then $I_\beta := I_{e_\beta} = \{f \in \mathbb{R}^\gamma_{\mathrm{lex}} : (\forall \alpha < \beta) \quad f(\alpha) = 0\}$
- All nonzero ideals of $\mathbb{R}^\gamma_{\mathrm{lex}}$ are of this form
- If $\alpha < \beta$, then $I_\beta \subsetneq I_\alpha$
- The set $I(\mathbb{R}^\gamma_{\mathrm{lex}})$ of ideals of $\mathbb{R}^\gamma_{\mathrm{lex}}$ ordered by inclusion is order anti-isomorphic with $\gamma + 1$
- Increasing sequences of ideals correspond to decreasing sequences in $\gamma + 1$, which stabilize since $\gamma + 1$ is well-ordered.

Goal: generalize $\mathbb{R}^\gamma_{\text{lex}}$ to a class containing Archimedean $\mathbb{R}^n$. For this we have to drop the requirement on γ that it is totally ordered, while somehow keeping the requirement that it is well-ordered.

Definition

Let P be a partially ordered set (poset). Then P is *well-founded* if every nonempty subset has a minimal element.

- Let P be a well-founded poset.
- Vector space: $\mathbb{R}^P_{\text{lex}} = \{f : P \to \mathbb{R}\}$
- Ordered by cone C of functions f that are positive in **all** minimal elements of its support.
- $\mathbb{R}^P_{\text{lex}}$ is a partially ordered vector space
- $\mathbb{R}^n \cong \mathbb{R}^P_{\text{lex}}$ with P the poset of n incomparable elements.

Questions: When is $\mathbb{R}^P_{\text{lex}}$ a vector lattice? When is it Noetherian?

Let P be a well-founded poset.

Definition

- P is a *forest* if incomparable elements have no common upper bound
- $A \subset P$ is an *antichain* if no elements in A are comparable
- P has *finite width* if every antichain is finite

Theorem

$\mathbb{R}^P_{\text{lex}}$ *is a vector lattice iff P is a forest.*

Theorem

Let P be a forest. Then $\mathbb{R}^P_{\text{lex}}$ is Noetherian iff P is a well-founded forest with finite width.

Lex(P)

Let P be a well-founded forest with finite width.

- Define Lex(P) to be the functions in $\mathbb{R}^P_{\mathrm{lex}}$ with finite support
- For $x \in P$, define $e_x(y) := \delta_{xy}$, then $e_x \in \mathrm{Lex}(P)$
- Lex(P) is the smallest sublattice of $\mathbb{R}^P_{\mathrm{lex}}$ containing $\{e_x : x \in P\}$.

Proposition

Any sublattice of $\mathbb{R}^P_{\mathrm{lex}}$ containing Lex(P) has the same ideals as $\mathbb{R}^P_{\mathrm{lex}}$.

Corollary

Let P be a well-founded forest with finite width. Then any sublattice of $\mathbb{R}^P_{\mathrm{lex}}$ containing Lex(P) is Noetherian.

Goal:

Theorem

Let E be a Noetherian vector lattice. Then there exists a well-founded forest with finite width P such that E is isomorphic to a sublattice of $\mathbb{R}^P_{\text{lex}}$ containing $\text{Lex}(P)$.

For the proof:

- How to find P?
- How to show that E is isomorphic to a sublattice of $\mathbb{R}^P_{\text{lex}}$ containing $\text{Lex}(P)$?

Local vector lattices

Again inspired by ring theory:

Definition

A vector lattice E is called *local* if it is unital and has a unique maximal ideal M.

Theorem

For a vector lattice E, the following are equivalent:

1. *E is local*
2. *E contains an ideal M and an element $e \in E \setminus M$ such that $E = \{\lambda e + u \colon \lambda \in \mathbb{R},\ u \in M\}$ and $\lambda e + u \in E_+$ iff $\lambda > 0$ or $\lambda = 0$ and $u \in M_+$.*
3. *$E \cong \mathbb{R} \circ M$*

Local ideals

Definition

An ideal L in a vector lattice is called local if it is a local vector lattice.

Example:

- P well-founded forest with finite width
- E sublattice of $\mathbb{R}^P_{\text{lex}}$ containing $\text{Lex}(P)$
- For $x \in P$, the ideal $L_x = \{f \in E \colon \text{supp}(f) \subset \{y \geq x\}\}$ is local with unit e_x and maximal ideal $M = \{f \in E \colon \text{supp}(f) \subset \{y > x\}\}$.
- These are the only local ideals in E
- So P is in bijection with the set of local ideals $L(E)$ of E

Semi-local

Definition

A vector lattice E (or an ideal of a vector lattice) is called *semi-local* if it is unital and has $1 \leq n < \infty$ maximal ideals.

Theorem

E is semi-local iff there are local ideals $L_1, \ldots, L_n$ such that

$$E \cong \bigoplus_{i=1}^{n} L_i \cong \bigoplus_{i=1}^{n} (\mathbb{R} \circ M_i).$$

Theorem

If E is Noetherian, then E is semi-local.

Proof.

Noetherian $\Rightarrow \operatorname{codim}(\operatorname{Rad}(E)) < \infty$.
(Schaefer) $\operatorname{codim}(\operatorname{Rad}(E)) < \infty \Rightarrow$ semi-local. □

Goal:

Theorem

Let E be a Noetherian vector lattice. Then there exists a well-founded forest with finite width P such that E is isomorphic to a sublattice of $\mathbb{R}^P_{\mathrm{lex}}$ containing $\mathrm{Lex}(P)$.

- Let P be the set of local ideals of E ordered by reverse inclusion
- For each $L \in P$, pick a unit e_L
- Idea: construct a lattice homomorphism $T\colon E \to \mathbb{R}^P_{\mathrm{lex}}$ with $Te_L(J) = \delta_{LJ}$ (J local)
- For $u \in E$, how to construct $Tu(L)$?
- If $u \in L \cong \mathbb{R} \circ M$, then $u = \lambda e_L + v$ and we can define $\varphi_L(u) := \lambda$, and so we can set $Tu(L) = \varphi_L(u)$
- In general $u \notin L$ so we have to construct a map from E into L

Theorem

Let E be a Noetherian vector lattice. Then there exists a collection of idempotents $\{Q_L \colon E \to E\}_{L \in L(E)}$ with $Q_L(E) = L$ such that:

- *If $L_1 \cap L_2 = \{0\}$, then $Q_{L_1} Q_{L_2} = 0$*
- *If $L_1 \subset L_2$, then $Q_{L_1} Q_{L_2} = Q_{L_1}$*

Now we can define $T \colon E \to \mathbb{R}^P_{\text{lex}}$ by $Tu(L) := \varphi_L(Q_L u)$, and this can be shown to be a lattice homomorphism.

- How to find the idempotents Q_L?

Constructing Q_L

Recall that Noetherian implies semi-local.

- $E \cong \oplus_{i=1}^{n} L_i$
- Define Q_{L_i} to be the band projection onto L_i
- $L_i \cong \mathbb{R} \circ M_i$, so define $Q_{M_i} := (\lambda e_{L_i} + v \mapsto v) \circ Q_{L_i}$
- M_i is Noetherian, so $M_i \cong \oplus_{k=1}^{n} L_{ik}$
- Define $Q_{L_{ik}}$ to be the composition of Q_{M_i} with the band projection of M_i onto L_{ik}
- $L_{ik} \cong \mathbb{R} \circ M_{ik}$, so define $Q_{M_{ik}} := (\lambda e_{L_{ik}} + v \mapsto v) \circ Q_{L_{ik}}$
- ... and so on...

Problem: in $\mathbb{R}^{\gamma}_{\text{lex}}$, this procedure only obtains Q_{L_n} for $n < \omega$ but not Q_{L_α} for $\alpha \geq \omega$. Something is needed to go "infinity and beyond".

Well-founded recursion

- Transfinite recursion: technique for constructing objects O_α for every ordinal α
- Example: Closure in convergence structures is obtained by "taking adherences transfinitely many times"
- Transfinite recursion can be easily generalized to well-founded posets (instead of well-ordered)
- The poset of ideals of a Noetherian vector lattice equipped with reverse inclusion is well-founded
- Well-founded recursion allows us to go "infinity and beyond", obtaining the desired Q_L for **all** local ideals L

Summary

In this presentation:

Theorem

Let E be a vector lattice. Then E is Noetherian if and only if there exists a well-founded poset P with finite width such that E is isomorphic to a sublattice of $\mathbb{R}^P_{\text{lex}}$ containing $\text{Lex}(P)$.

We also have a full characterization of Artinian vector lattices:

Theorem

Let E be a vector lattice. Then E is Artinian if and only if there exists a reverse well-founded poset P with finite width such that $E \cong \text{Lex}(P)$.

Thank you for your attention!