Prime ideals and Noetherian properties in vector lattices

Mark Roelands, joint work with Marko Kandić

Positivity XI

July 13, 2023

Prime ideals

A proper ideal P in E is called *prime* if one of the following equivalent conditions holds

- If $x \wedge y \in P$, then either $x \in P$ or $y \in P$.
- If $x \wedge y = 0$, then either $x \in P$ or $y \in P$.
- E/P is totally ordered.

Prime ideals

A proper ideal P in E is called *prime* if one of the following equivalent conditions holds

- If $x \wedge y \in P$, then either $x \in P$ or $y \in P$.
- If $x \wedge y = 0$, then either $x \in P$ or $y \in P$.
- E/P is totally ordered.

Some facts

- Every maximal ideal in E is prime.

Prime ideals

A proper ideal P in E is called *prime* if one of the following equivalent conditions holds

- If $x \wedge y \in P$, then either $x \in P$ or $y \in P$.
- If $x \wedge y = 0$, then either $x \in P$ or $y \in P$.
- E/P is totally ordered.

Some facts

- Every maximal ideal in E is prime.
- Every proper ideal containing a prime ideal is prime.

Prime ideals

A proper ideal P in E is called *prime* if one of the following equivalent conditions holds

- If $x \wedge y \in P$, then either $x \in P$ or $y \in P$.
- If $x \wedge y = 0$, then either $x \in P$ or $y \in P$.
- E/P is totally ordered.

Some facts

- Every maximal ideal in E is prime.
- Every proper ideal containing a prime ideal is prime.
- The collection of ideals containing a fixed prime ideal is totally ordered by set inclusion.

Lemma (Prime avoidance lemma)

Let I be a proper ideal in E and $x \in E \setminus I$. Then an ideal that is maximal w.r.t. not containing x and containing I is prime.

Prime ideals

Lemma (Prime avoidance lemma)

Let I be a proper ideal in E and $x \in E \setminus I$. Then an ideal that is maximal w.r.t. not containing x and containing I is prime.

Note: all nonzero vector lattices contain prime ideals.

Prime ideals

Examples

- In $c_{00}(\Omega)$, the prime ideals are precisely of the form

$$P_\omega := \{x \in c_{00}(\Omega) \colon x(\omega) = 0\}.$$

Hence, all prime ideals in $c_{00}(\Omega)$ are maximal.

Examples

- In $c_{00}(\Omega)$, the prime ideals are precisely of the form

$$P_\omega := \{x \in c_{00}(\Omega) \colon x(\omega) = 0\}.$$

  Hence, all prime ideals in $c_{00}(\Omega)$ are maximal.
- Any maximal multiplicatively closed set F not containing 0 in $C(K)$ yields an algebraic prime ideal $C(K) \setminus F$.

Prime ideals

Theorem

Let E be an Archimedean vector lattice with finitely many prime ideals. Then it must be finite dimensional.

Prime ideals

Theorem

Let E be an Archimedean vector lattice with finitely many prime ideals. Then it must be finite dimensional.

Idea

- Any principal ideal I_x contains finitely many prime ideals.

Prime ideals

Theorem

Let E be an Archimedean vector lattice with finitely many prime ideals. Then it must be finite dimensional.

Idea

- Any principal ideal I_x contains finitely many prime ideals.
- A norm dense sublattice of $C(K)$ is finite dimensional if there are only finitely many prime ideals.

Prime ideals

Theorem

Let E be an Archimedean vector lattice with finitely many prime ideals. Then it must be finite dimensional.

Idea

- Any principal ideal I_x contains finitely many prime ideals.
- A norm dense sublattice of $C(K)$ is finite dimensional if there are only finitely many prime ideals.
- Now $E \cong c_{00}(\Omega)$ and Ω must be finite.

From commutative rings to vector lattices

A vector lattice E is said to be *Noetherian* if every ascending chain of ideals in E is stationary.

A vector lattice E is said to be *Noetherian* if every ascending chain of ideals in E is stationary.

Proposition

Let E be an Archimedean vector lattice. Then E is Noetherian if and only if E is finite dimensional.

From commutative rings to vector lattices

A vector lattice E is said to be *Noetherian* if every ascending chain of ideals in E is stationary.

Proposition

Let E be an Archimedean vector lattice. Then E is Noetherian if and only if E is finite dimensional.

Note: the assumption of E being Archimedean cannot be dropped, since $\mathrm{Lex}(\mathbb{N})$ is an infinite dimensional Noetherian vector lattice.

From commutative rings to vector lattices

A vector lattice E is said to be *Noetherian* if every ascending chain of ideals in E is stationary.

Proposition

Let E be an Archimedean vector lattice. Then E is Noetherian if and only if E is finite dimensional.

Note: the assumption of E being Archimedean cannot be dropped, since $\mathrm{Lex}(\mathbb{N})$ is an infinite dimensional Noetherian vector lattice.

Ideal structure is $\mathrm{Lex}(\mathbb{N}) = I_{e_1} \supsetneq I_{e_2} \supsetneq I_{e_3} \supsetneq \dots$ and $I_{e_k}/I_{e_{k+1}} \cong \mathbb{R}$.

From commutative rings to vector lattices

Theorem

Let R be a commutative ring.

- **Cohen:** *R is Noetherian if and only if every prime ideal is finitely generated.*

From commutative rings to vector lattices

Theorem

Let R be a commutative ring.

- **Cohen:** *R is Noetherian if and only if every prime ideal is finitely generated.*
- **Kaplansky:** *If R is Noetherian, every ideal is principal if and only if every maximal ideal is principal.*

From commutative rings to vector lattices

Theorem

Let R be a commutative ring.

- **Cohen:** *R is Noetherian if and only if every prime ideal is finitely generated.*
- **Kaplansky:** *If R is Noetherian, every ideal is principal if and only if every maximal ideal is principal.*
- **Cohen-Kaplansky:** *If every prime ideal is principal, then every ideal is principal.*

From commutative rings to vector lattices

Theorem

Let E be a vector lattice.

(i) *E is finite dimensional.*

(ii) *Every proper ideal in E is principal (same as finitely generated).*

(iii) *Every prime ideal in E is principal (same as finitely generated).*

From commutative rings to vector lattices

Theorem

Let E be a vector lattice.

- (i) *E is finite dimensional.*
- (ii) *Every proper ideal in E is principal (same as finitely generated).*
- (iii) *Every prime ideal in E is principal (same as finitely generated).*

Then $(i) \implies (ii)$, $(i) \implies (iii)$, $(iii) \implies (ii)$, and if E is Archimedean, then $(ii) \implies (i)$.

From commutative rings to vector lattices

Theorem

Let E be a vector lattice.

- (i) *E is finite dimensional.*
- (ii) *Every proper ideal in E is principal (same as finitely generated).*
- (iii) *Every prime ideal in E is principal (same as finitely generated).*

Then $(i) \implies (ii)$, $(i) \implies (iii)$, $(iii) \implies (ii)$, *and if E is Archimedean, then* $(ii) \implies (i)$.

Hence if E is Archimedean,

- then (i) and (iii) are equivalent (Cohen's theorem for vector lattices).
- the equivalence between (ii) and (iii) is the Cohen-Kaplansky theorem for vector lattices.

Prime Noetherian vector lattices

A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals is stationary.

Prime Noetherian vector lattices

A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals is stationary.

Theorem

Let E be a uniformly complete Archimedean vector lattice. Then E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$.

Prime Noetherian vector lattices

A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals is stationary.

Theorem

Let E be a uniformly complete Archimedean vector lattice. Then E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$.

Idea

- Being prime Noetherian passes down to ideals, so I_x is prime Noetherian.

Prime Noetherian vector lattices

A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals is stationary.

Theorem

Let E be a uniformly complete Archimedean vector lattice. Then E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$.

Idea

- Being prime Noetherian passes down to ideals, so I_x is prime Noetherian.
- If $C(K)$ is prime Noetherian, then it is finite dimensional.

Prime Noetherian vector lattices

A vector lattice E is said to be *prime Noetherian* if every ascending chain of prime ideals is stationary.

Theorem

Let E be a uniformly complete Archimedean vector lattice. Then E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$.

Idea

- Being prime Noetherian passes down to ideals, so I_x is prime Noetherian.
- If $C(K)$ is prime Noetherian, then it is finite dimensional.
- If every principal ideal is finite dimensional, $E \cong c_{00}(\Omega)$.

Prime Noetherian vector lattices

For $n \in \mathbb{N}$, denote by $\mathrm{PPol}^n([a, b])$ the vector lattice of continuous piecewise polynomials of degree at most n on $[a, b]$.

Prime Noetherian vector lattices

For $n \in \mathbb{N}$, denote by $\mathrm{PPol}^n([a,b])$ the vector lattice of continuous piecewise polynomials of degree at most n on $[a,b]$.

Further, denote by $\mathrm{PPol}([a,b])$ the vector lattice of continuous piecewise polynomials without a bound on the degree.

Prime Noetherian vector lattices

For $n \in \mathbb{N}$, denote by $\mathrm{PPol}^n([a, b])$ the vector lattice of continuous piecewise polynomials of degree at most n on $[a, b]$.

Further, denote by $\mathrm{PPol}([a, b])$ the vector lattice of continuous piecewise polynomials without a bound on the degree.

Theorem

In $\mathrm{PPol}^n([a, b])$ *every ascending chain of prime ideals has at most length* n*, and* $\mathrm{PPol}([a, b])$ *is prime Noetherian, and contains ascending chains of prime ideals of arbitrary finite length.*

Prime Noetherian vector lattices

Define $E \subsetneq C([0,1])$ by $f \in E$ if there is an $\varepsilon > 0$ such that for some $n \in \mathbb{N}$,

$$f(t) = a_0 + \sum_{k=1}^{n} a_k t^{\frac{1}{k}} \quad (t \in [0, \varepsilon)).$$

Prime Noetherian vector lattices

Define $E \subsetneq C([0,1])$ by $f \in E$ if there is an $\varepsilon > 0$ such that for some $n \in \mathbb{N}$,

$$f(t) = a_0 + \sum_{k=1}^{n} a_k t^{\frac{1}{k}} \quad (t \in [0,\varepsilon)).$$

In E we have the ascending chain of prime ideals

$$I_t \subsetneq I_{t^{1/2}} \subsetneq \cdots \subsetneq I_{t^{1/n}} \subsetneq I_{t^{1/(n+1)}} \subsetneq \cdots$$

so it is not prime Noetherian.

Paper and thanks

The paper can be found here:

Marko Kandić and Mark Roelands. Prime ideals and Noetherian properties in vector lattices, *Positivity*, 26(1), 13, 2022.

Paper and thanks

The paper can be found here:

Marko Kandić and Mark Roelands. Prime ideals and Noetherian properties in vector lattices, *Positivity*, 26(1), 13, 2022.

Thank you for your attention!