Relations between selected geometric properties on the positive cone of all nonnegative and decreasing elements of symmetric Banach spaces

Maciej Ciesielski

Institute of Mathematics
Poznan University of Technology, Poland

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GOAL

1. Preliminary
2. Rotundity properties and reflexivity
3. K-order continuity
4. Uniform K-monotonicity
5. Applications

H1 M. C., *Relationships between K-monotonicity and rotundity properties with application*, J. Math. Anal. Appl. **465** (2018), no. 1, 235-258.

H2 M. C., *Strict K-monotonicity and K-order continuity in symmetric spaces*, Positivity **22** (2018), no. 3, 727–743.

H3 M. C. and G. Lewicki, *Uniform K-monotonicity and K-order continuity in symmetric spaces with application to approximation theory*, J. Math. Anal. Appl. **456** (2017), no. 2, 705–730.

Let $I = [0, \alpha)$, where $\alpha = \infty$ or $\alpha = 1$ and μ be the **Lebesgue measure** on $\mathbb{R}$.

Let $L^0 = L^0(I, \mu)$ be a space of all classes of $f : I \to \overline{\mathbb{R}}$ μ**-measurable** extended real-valued functions.

(Quasi-) Banach ideal space E, is a linear subspace of L^0, equipped with a complete quasi-norm $\|\cdot\|_E : E \to \mathbb{R}_+$ satisfying:

(*i*) If $f \in L^0$, $g \in E$ and $|f| \leq |g|$ a.e., then $f \in E$, $\|f\|_E \leq \|g\|_E$,

(*ii*) There exists a **weak unit**, i.e. strictly positive $f \in E$.

Denote

$$B_E = \{x \in E : \|x\|_E \leq 1\} \quad \text{i} \quad S_E = \{x \in E : \|x\|_E = 1\}.$$

We assume that E has **Fatou property**, i.e. for any $(x_n) \subset E^+ = \{x \geq 0 : x \in E\}$,

$$\sup_{n \in \mathbb{N}} \|x_n\|_E < \infty \quad \text{and} \quad x_n \uparrow x \in L^0 \quad \text{a.e.} \quad \Rightarrow \quad x \in E \quad \text{and} \quad \|x_n\|_E \uparrow \|x\|_E .$$

A point $x \in E$ is a **point of order continuity** ($x \in E_a$) if for any $(x_n) \subset E^+$,

$$x_n \leq |x| \quad \text{and} \quad x_n \to 0 \quad \text{a.e.} \quad \Rightarrow \quad \|x_n\|_E \to 0.$$

E is called **order continuous** ($E \in (OC)$) if and only if $E = E_a$.

For any Banach ideal space E we define the **associate space** E' by

$$E' = \left\{ y \in L^0 : \|y\|_{E'} = \sup\left\{ \int_I |xy| \, d\mu : \|x\|_E \leq 1 \right\} < \infty \right\}.$$

The **distribution function** of any $x \in L^0$

$$d_x(\lambda) = \mu\left(\{t \in I : |x(t)| > \lambda\}\right) \qquad \text{for all} \quad \lambda \geq 0.$$

The **decreasing rearrangement** of $x \in L^0(I)$

$$x^*(t) = \inf\left\{s > 0 : d_x(s) \leq t\right\}, \quad \text{for all} \quad t > 0.$$

The **maximal function** of x^*

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds, \quad \text{for all} \quad t > 0.$$

The **Hardy-Littlewood-Pólya relation** $\prec$ is given for any $x, y \in L^1 + L^\infty$ by

$$x \prec y \quad \Leftrightarrow \quad x^{**}(t) \leq y^{**}(t) \quad \text{for all} \quad t > 0.$$

We assume

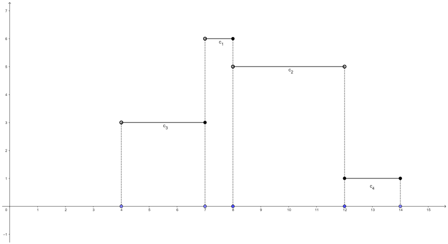
$$E^+ = \{x \geq 0 : x \in E\}, \qquad E^d = \{x^* : x \in E\}$$

and

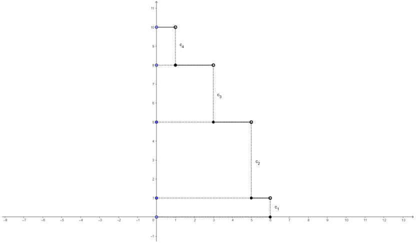
$$x^*(\infty) = \lim_{t \to \infty} x^*(t) \quad \text{if } \alpha = \infty \quad \text{and} \quad x^*(\infty) = 0 \quad \text{if } \alpha = 1.$$

The distribution function d_x and decreasing rearrangement x^* of the simple function x

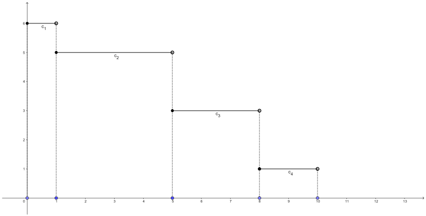
the simple function x



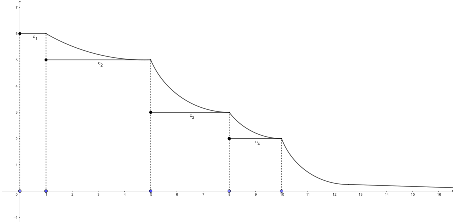
the distribution d_x



the decreasing rearrangement x^*



the maximal function x^{**}



(Quasi-)Banach ideal space $(E, \|\cdot\|_E)$ is said to be **symmetric (quasi-)Banach space**, if for any $x \in L^0$ and $y \in E$

$$d_x(\lambda) = d_y(\lambda) \quad \text{for any } \lambda > 0 \quad \Rightarrow \quad x \in E \quad \text{and} \quad \|x\|_E = \|y\|_E .$$

Symmetric Banach space we call shortly **symmetric space**.

For any symmetric quasi-Banach space E we define **fundamental function** ϕ_E by

$$\phi_E(t) = \left\|\chi_{(0,t)}\right\|_E \quad \text{for any} \quad t \in I.$$

A Banach ideal space E is called **reflexive**, if for any linear and continuous functional $f \in E^*$

there exists $x \in S_E$ such that $f(x) = \|f\|_{E^*}$.

Let $k \in \mathbb{N}$, $k \geq 2$. A Banach ideal space E is called **fully k-rotund**, (**compactly fully k-rotund**), if each $(x_n) \subset S_E$ such that

$$\left\| \sum_{i=1}^{k} x_{n,i} \right\|_E \to k \quad \text{for any its } k\text{-subsequences } (x_{n,1}), (x_{n,2}), \ldots, (x_{n,k}),$$

is a Cauchy sequence (**forms a relatively compact set**).
A point $x \in S_E$ is said to be a **point of local fully k-rotundity**, if for each $(x_n) \subset S_E$ such that

$$\left\| x + \sum_{i=1}^{k} x_{n,i} \right\|_E \to k+1 \quad \text{for any its } k\text{-subsequences} \quad (x_{n,1}), (x_{n,2}), \cdots, (x_{n,k}),$$

we have x_n converges to x in E.
A Banach ideal space E is called **locally fully k-rotund**, if every $x \in S_E$ is a point of local fully k-rotundity.

- K. Fan i I. Glicksberg, 1955
  Introduced **fully k-rotundity** and **compact fully k-rotundity** and presented equivalent characterizations for given properties.
- Y. Cui, H. Hudzik and W. Kowalewski, 2003
  Proved strict relation between **fully k-rotundity**, **compact fully k-rotundity** and **strict convexity** in Banach spaces.
- H. Hudzik, W. Kowalewski, and G. Lewicki, 2006
  Presented a complete criteria for **fully 2-rotundity** in Orlicz-Lorentz spaces with Luxemburg norm.

Theorem 2.1 (H1, Theorem 4.5)

Let E be a symmetric space. The following conditions are equivalent.

(i) *x is **point of local fully k-rotundity**.*

(ii) *$|x|$ is **point of local fully k-rotundity**.*

(iii) *x^* is **point of local fully k-rotundity**.*

Theorem 2.2 (H1, Theorem 4.7)

Let E be a symmetric space and let

(i) *E is **locally fully k-rotund**.*

(ii) *E^+ is **locally fully k-rotund**.*

(iii) *E^d is **locally fully k-rotund**.*

Then,

$$(i) \Leftrightarrow (ii) \Rightarrow (iii).$$

If E is order continuous, then

$$(i) \Leftarrow (iii).$$

Relations between rotundity properties and reflexivity

- D.P. Milman, B.J. Pettis, 30-ties
  Proved that **uniform rotundity** implies **reflexivity** in Banach spaces.
- M.M. Day, 1941
  Presented an example of **reflexive** Banach space that does not have an equivalent norm that is **uniformly rotund**
- K. Fan, I. Glicksberg, 1958
  Fully k**-rotund** Banach space X is **reflexive**
- E. Asplund, 1968
  Relation between **reflexivity** of a Banach space X and an existence of an equivalent norm in X, for which the dual space X^* is **locally uniformly rotund**.
- Y. Cui, H. Hudzik and W. Kowalewski, 2003
  Compact fully k**-rotund** Banach space X is **reflexive**.

Theorem 2.3 (H1, Theorem 5.4)

Let E be a symmetric space.
*If $E^d = \{x^* : x \in E\}$ is **compact fully** k**-rotund**, then E is **reflexive**.*

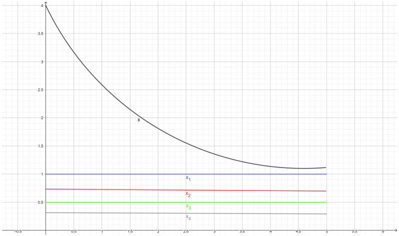
$x \in E$ is a **point of K-order continuity** in a symmetric quasi-Banach space E if for any $(x_n) \subset E$,

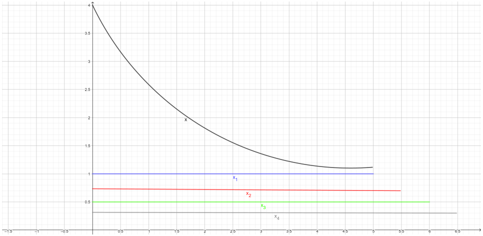
$$x_n \prec x \quad \text{and} \quad x_n^* \to 0 \quad \text{a.e.} \quad \Rightarrow \quad \|x_n\|_E \to 0.$$

E is K-**order continuous**, $E \in (KOC)$, if any $x \in E$ is a point of K-order continuity.

Difference between order continuity and K-order continuity

$$x_n \leq x \qquad\qquad x_n \prec x \quad \left(\int_0^t x_n^* \leq \int_0^t x^*\right)$$





- P.G. Dodds, E.M. Semenov, F.A. Sukochev, 2004
  Initiated research devoted to *KOC* property for order continuous symmetric spaces with the Fatou property.
- P.G. Dodds, T.K. Dodds, F.A. Sukochev, 2007
  Investigated *KOC* in the more general setting of symmetric spaces of measurable operators under additional assumption.
- Originally *KOC* was introduced by an equivalent definition i.e. for any $x \in E$, $(x_n) \subset E$,

$$x_n \prec x \quad \text{and} \quad x_n \to 0 \quad \text{globally in measure} \quad \Rightarrow \quad \|x_n\|_E \to 0.$$

Theorem 3.1 (H2, Theorem 2)

Let E be a symmetric space with the fundamental function ϕ and let $x \in E$. The following conditions are equivalent.

(*i*) *x is a point of order continuity and* $\lim_{s\to\infty} \phi(s)x^{**}(s) = 0$.

(*ii*) *x is a* ***point of K-order continuity*** *and* $x^*(\infty) = 0$.

If we omit the assumption $x^*(\infty)=0$ in (ii) or $\lim_{s\to\infty}\phi(s)x^{**}(s)=0$ in (i), then the set of points of KOC is distinct from the set of points of OC.

Example 3.2 (H2, Example 1, Remark 1)

In Marcinkiewicz space

$$M_\phi^{(*)}=\left\{x\in L^0:\sup_{t>0}\{x^{**}(t)\phi(t)\}<\infty\right\},\quad \text{where}\quad \phi(t)=1-\frac{1}{1+t}\quad \text{for } t\in[0,\infty),$$

*there exists $x=\chi_{(0,\infty)}\in M_\phi^{(*)}$ a **point of** KOC, that is not a **point of** OC.*

Lorentz space $\Lambda_{1,\phi'}$, where ϕ is strictly concave fundamental function such that

$$\phi(0^+)=0\quad \text{i}\quad \phi(\infty)=\infty\quad \text{oraz}\quad \sup_{t>0}\frac{t}{\phi(t)}<\infty,$$

*is **order continuous** symmetric space and does not contain **points of** KOC.*

Theorem 3.3 (H2, Theorem 3.3)

Let E be a symmetric space with the fundamental function ϕ on $I=[0,\infty)$. The following conditions are equivalent.

- *(i) E is **order continuous** and is not embedded in $L^1[0,\infty)$.*
- *(ii) E is **order continuous** and $E'\hookrightarrow\{f:f^*(\infty)=0\}$.*
- *(iii) E is K**-order continuous** and $\phi(\infty)=\infty$.*

A symmetric quasi-Banach space E is said to be:

- **upper locally uniformly K-monotone**, if for any $x \in E$, $(y_n) \in E$,

$$x \prec y_n, \qquad \lim_{n\to\infty} \|y_n\|_E = \|x\|_E \qquad \Rightarrow \qquad \lim_{n\to\infty} \|x^* - y_n^*\|_E = 0.$$

- **uniformly K-monotone**, $E \in (UKM)$, if for any $(x_n), (y_n) \subset E$,

$$x_n \prec y_n, \qquad \lim_{n\to\infty} \|x_n\|_E = \lim_{n\to\infty} \|y_n\|_E \qquad \Rightarrow \qquad \lim_{n\to\infty} \|x_n^* - y_n^*\|_E = 0.$$

- **decreasing uniformly K-monotone,** $E \in (DUKM)$, if for any $(x_n), (y_n) \subset E$,

$$x_{n+1} \prec x_n \prec y_n, \qquad \lim_{n\to\infty} \|x_n\|_E = \lim_{n\to\infty} \|y_n\|_E \qquad \Rightarrow \qquad \lim_{n\to\infty} \|x_n^* - y_n^*\|_E = 0.$$

- **increasing uniformly K-monotone,** $E \in (IUKM)$, if for any $(x_n), (y_n) \subset E$,

$$x_n \prec y_n \prec y_{n+1}, \qquad \lim_{n\to\infty} \|x_n\|_E = \lim_{n\to\infty} \|y_n\|_E \qquad \Rightarrow \qquad \lim_{n\to\infty} \|x_n^* - y_n^*\|_E = 0.$$

Uniform K-monotonicity was introduced in [H3]:

- Generalization of **uniform monotonicity**, $E \in (UM)$, i.e. for any $(x_n), (y_n) \subset E^+$,

$$x_n \leq y_n, \qquad \lim_{n\to\infty} \|x_n\|_E = \lim_{n\to\infty} \|y_n\|_E \qquad \Rightarrow \qquad \|x_n - y_n\|_E \to 0.$$

- Describes the different geometric structure of a space, e.g. $L^1[0,\infty) \in (UM)$, $\notin (UKM)$.
- Restriction of **uniform rotundity** to comparable pairs of elements in the sens of H-L-P relation $\prec$ on the positive cone E^d.
- Crucial tool for solving a **problem of the best approximation** on the K-directed sets in the sens of the relation $\prec$ in symmetric spaces.

Theorem 3.4 (H2, Theorem 3.4)

Let E be a symmetric space with the fundamental function ϕ, where

$$\phi(\infty) = \infty \quad \text{if} \quad I = [0, \infty).$$

Then, the following conditions are equivalent.

(*i*) *E is **decreasing uniformly K-monotone**.*

(*ii*) *E is K**-order continuous** and **upper locally uniformly K-monotone**.*

A symmetric space E is called **strict K-monotone**, $E \in (SKM)$, if for any $x, y \in E$,

$$x^* \neq y^*, \quad x \prec y \quad \Rightarrow \quad \|x\|_E < \|y\|_E .$$

Theorem 3.5 (H2, Theorem 5.7 & Theorem 5.8)

Let E be a symmetric space.
*If $E^d = \{x^* : x \in E\}$ is **compactly fully k-rotund** and **strictly K-monotone**, then*
*E is **decreasing uniformly K-monotone** and **increasing uniformly K-monotone**.*

Let A be a subset of a Banach space X and let $f \in X$. A mapping

$$P_A(f) = \left\{ a \in A : \|f - a\|_X = \inf_{b \in A} \|f - b\|_X \right\}$$

is called a **metric projection** or a **best approximant operator**.

A subset A in a symmetric space E is said to be K**-directed set** if for any $x, y \in A$ there exists $z \in A$ such that

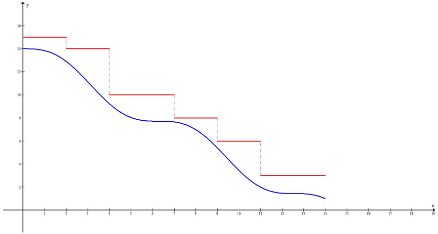
$$z \prec x \quad \text{and} \quad z \prec y.$$

For each $A \subset E$ we define

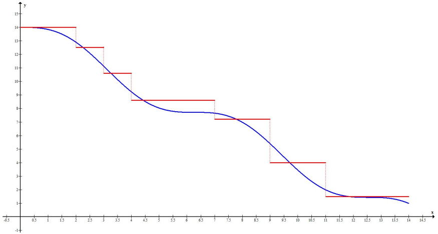
$$K(A) = \{x \in E : A - x \text{ is } K\text{-directed set}\}.$$

Difference between the best dominated approximation with respect to $\prec$ and $\leq$

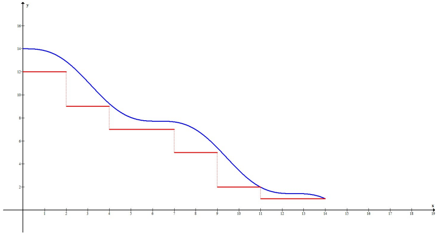
$f \leq g$



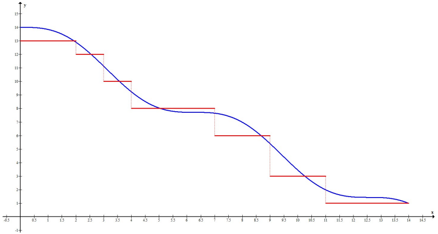
$f \prec g \quad (\int_0^t f \leq \int_0^t g)$



$g \leq f$



$g \prec f \quad (\int_0^t g \leq \int_0^t f)$



Continuity of the best dominated approximant operator with respect to $\prec$

- K. Fan i I. Glicksberg, 60-ties
  Metric projection P_A given on any closed and convex set A, in **strictly convex** and **reflexive** Banach space X with the **Kadeca-Klee property**, is continuous.
- A.L. Brown, 80-ties
  If **Kadeca-Klee property** is avoided in assumptions of X, then the metric projection P_A is not continuous.
- H. Hudzik, W. Kowalewski, G. Lewicki, 2006
  Reflexivity and **Kadeca-Klee property** $\Leftrightarrow$ **approximative compactness** in a Banach space X, e.a. for any nonempty closed convex subset $C \subset X$ and for any $x \in X$, $(x_n) \subset C$,

$$\|x_n - x\|_X \to d(x, C) = \inf_{y \in C} \|y - x\|_X \quad \Rightarrow \quad (x_n) \text{ has a Cauchy subsequence.}$$

Theorem 4.1 (H3, Theorem 5.19)

Let E be a symmetric space, $A \subset E$ be a convex subset, $f_n \in K(A)$ with $P_A(f_n) = \{x_n\}$ for $n \in \mathbb{N}$. If E is **upper locally uniformly** *K***-monotone** *and* **order continuous**, *then P_A is continuous,*

$$\|f_n - f_0\|_E \to 0 \quad \Rightarrow \quad \|P_A(f_n) - P_A(f_0)\|_E = \|x_n - x_0\|_E \to 0.$$

We investigate a stronger version of the above problem for ordered sets. Indeed, in Banach ideal space with Fatou property, **reflexivity** implies **order continuity**. In order continuous symmetric spaces, **Kadeca-Klee property** implies **upper local uniform** K**-monotonicity**.

Stability of the best approximation problem with respect to $\prec$

A sequence $(x_n) \subset A - f$ is called a **minimizing sequence** in $A - f$, if

$$\lim_{n\to\infty} \|x_n\|_E = \inf_{a\in A} \|f - a\|_E = \operatorname{dist}(f, A).$$

The best approximation problem is **stable**, if for any minimizing sequence $(x_n) \subset A - f$ we have

$$\operatorname{dist}(x_n + f, P_A(f)) \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 4.2 (H3, Theorem 5.7)

Let E be a symmetric space. The following conditions are equivalent.

(i) *E is* **upper locally uniformly K-monotone**.

(ii) *For any convex subset $A \subset E$ and for any $f \in K(A)$,*

$$(x_n) \subset A - f \quad \text{is minimizing sequence and} \quad x + f \in P_A(f) \quad \Rightarrow \quad \|x_n^* - x^*\|_E \to 0.$$

(iii) *For any closed convex subset $A \subset E$ and for any $f \in K(A)$,*

$$(x_n) \subset A - f \quad \text{is minimizing sequence and} \quad x + f \in P_A(f) \quad \Rightarrow \quad \|x_n^* - x^*\|_E \to 0.$$

Decreasing uniform K-monotonicity and the best dominated approximation problem

Theorem 4.3 (H3, Theorem 5.22)

Let E be a symmetric space. The following conditions are equivalent.

- (i) *E is **decreasing uniformly K-monotone**.*
- (ii) *For any K-directed subset A of E and for any minimizing sequence (x_n) of A we have (x_n^*) is Cauchy.*
- (iii) *For any convex K-directed subset A of E and for any minimizing sequence (x_n) of A we have (x_n^*) is Cauchy.*

Thank You for your attention.

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