On different modes of Order Convergence

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July 11, 2023

Definition 1 (Different modes of order convergence)

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net and x a point in a poset P.

1. $(x_\gamma)_{\gamma\in\Gamma}$ is said to O_1-*converge* to x in P if there exist two nets $(y_\gamma)_{\gamma\in\Gamma}$, $(z_\gamma)_{\gamma\in\Gamma}$ in P such that eventually $y_\gamma \leq x_\gamma \leq z_\gamma$, $y_\gamma \uparrow x$ and $z_\gamma \downarrow x$.
2. $(x_\gamma)_{\gamma\in\Gamma}$ is said to O_2-*converge* to x in P if there exists a directed subset $M \subset P$, and a filtered subset $N \subset P$, such that $\vee M = \wedge N = x$, and for every $(m, n) \in M \times N$ the net is eventually contained in $[m, n]$.
3. $(x_\gamma)_{\gamma\in\Gamma}$ is said to O_3-*converge* to x in P if there exist two subsets M and N of P such that $\vee M = \wedge N = x$, and for every $(m, n) \in M \times N$ the net is eventually contained in $[m, n]$.

For a net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P,

$$x_\gamma \xrightarrow{O_1} x \;\Rightarrow\; x_\gamma \xrightarrow{O_2} x \;\Rightarrow\; x_\gamma \xrightarrow{O_3} x \qquad (x \in \mathcal{P}).$$

Every eventually constant net is O_1-convergent to its eventual value.

Every cofinal subnet of an O_i-convergent net is O_i-convergent to the same limit (where $i \in \{1, 2, 3\}$)

When $(x_\gamma)_{\gamma\in\Gamma}$ is a net in a *Dedekind complete lattice*, the expression '$(x_\gamma)_{\gamma\in\Gamma}$ is O_i-convergent to x' conveys the intuitive meaning

$$\liminf_\gamma x_\gamma = x = \limsup_\gamma x_\gamma.$$

The following example shows that, even when the poset is a Boolean algebra or a Banach lattice, O_2-convergence does not imply O_1-convergence.

Example 2 (Fremlin)

Let X be an uncountable set, $\mathscr{D}$ the cofinite filter in X and $\mathscr{A}$ the Boolean algebra of all finite or cofinite subsets of X. Let (x_n) be a sequence of distinct elements of X and $A_n := \{x_n\}$.

Clearly, $\mathscr{D} \downarrow \emptyset$ in $\mathscr{A}$, and therefore (A_n) O_2-converges to $\emptyset$. On the other-hand, any O_1-conv. seq. in $\mathscr{A}$ is eventually constant.

Endow X with the discrete topology and let X_0 be the one-point compactification of X, i.e. $X_0 = X \cup \{\infty\}$, equipped with the topology $\mathscr{T} := 2^X \cup \hat{\mathscr{D}}$, where $\hat{\mathscr{D}} := \{D \cup \{\infty\} : D \in \mathscr{D}\}$. Then $(\chi_D)_{D \in \hat{\mathscr{D}}} \downarrow 0$ in $C(X_0, \mathbb{R})$ and therefore χ_{A_n} O_2-converges to 0.

On the other-hand, if $f \in C(X_0, \mathbb{R})$ and $f(x) \geq 1$ holds on an infinite subset of X_0 then, for every $\varepsilon > 0$, there exists $D \in \mathscr{D}$ s.t. $f(x) > 1 - \varepsilon$ on D. So, if $(g_n)_{n \in \mathbb{N}}$ is a decreasing sequence in $C(X_0, \mathbb{R})$ satisfying $\chi_{A_n} \leq g_n$ for every $n \in \mathbb{N}$, then the zero function cannot be the infimum of $(g_n)_{n \in \mathbb{N}}$. Hence $(\chi_{A_n})_{n \in \mathbb{N}}$ cannot be O_1-convergent to 0.

This following example shows that in general O_3-convergence does not imply O_2-convergence.

Example 3 (E. S. Wolk)

Let $A := \{a_n : n \in \mathbb{N}\}$ and $B := \{b_n : n \in \mathbb{N}\}$ be two countable sets and let $P := A \cup B \cup \{0, 1\}$ equipped with the partial order defined by:

$$0 \leq x \leq 1 \ (\forall x \in P) \text{ and } x \leq y \ \Leftrightarrow \ x = a_n,\ y = b_m \ (\forall n \leq m).$$

It is clear that P contains no infinite directed (or filtered) set. On the other-hand, $\sup A = 1$. Thus, the sequence $(b_n)_{n\in\mathbb{N}}$ is O_3-convergent to 1 but not O_2-convergent.

For a partially ordered set P, let $\hat{P}$ be the Dedekind-MacNeille completion of P and φ the order-embedding of P into $\hat{P}$.

Definition 4

A net $(x_\gamma)_{\gamma\in\Gamma}$ in a poset P is said to *O^{DM}-converge* to $x \in P$ if the net $\left(\varphi(x_\gamma)\right)_{\gamma\in\Gamma}$ O-converges to $\varphi(x)$ in $\hat{P}$.

Theorem 5

1. *In a poset, O_3-convergence is equivalent to O^{DM}-convergence.*
2. *In a lattice, O_2-convergence, O_3-convergence and O^{DM}-convergence are equivalent.*

Theorem 6 (J. C. Mathews and R. F. Anderson)

In a monotone order separable poset, O_2-convergence implies O_1-convergence.

Theorem 7

If a net $(x_\gamma)_{\gamma\in\Gamma}$ of poset P is O_2-convergent to $x \in P$, then $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that O_1-converges to x.

1. A subset X of a poset P is said to be *O_i-closed* if there is no net in X O_i-converging to a point outside of X for $i \in \{1, 2, 3\}$.
2. The collection of O_i-closed sets comprises the closed sets for a topology, which we shall denote by $\tau_{O_i}(P)$.

Since O_1-convergence is 'stronger' than O_2-convergence, and the later is yet 'stronger' than O_3-convergence, it follows that

$$\tau_{O_3}(P) \subset \tau_{O_2}(P) \subset \tau_{O_1}(P).$$

When P is a Dedekind complete lattice, all three order topologies are equal; in this case we write $\tau_O(P)$

Theorem 8

Let $(P, \leq)$ be a poset and let $\hat{P}$ denote its Dedekind-MacNeille completion.

1. $\tau_{O_1}(P) = \tau_{O_2}(P)$.
2. $\tau_{O_3}(P) \supset \tau_O(\hat{P})|_P$.
3. *If P is a lattice,* $\tau_{O_1}(P) = \tau_{O_2}(P) = \tau_{O_3}(P)$.

A lattice constructed by V. Olejček was verified to show that the inclusion $\tau_{O_3}(P) \supset \tau_O(\hat{P})|_P$ can indeed be proper.

A function $f : P \to Q$, where P and Q are two posets, is said to *preserve* O_i*-convergence* if $(f(x_\gamma))_{\gamma\in\Gamma}$ O_i-converges to $f(x)$ in Q whenever $(x_\gamma)_{\gamma\in\Gamma}$ is a net in P that O_i-converges to x.

Proposition 9 (AbramovichSirotkin2005)

If T is a linear operator from a Riesz space E into a Riesz space F, then

$$T \text{ preserves } \mathrm{O}_1\text{-convergence} \;\Rightarrow\; T \text{ preserves } \mathrm{O}_2\text{-convergence} .$$

When T is isotone, T preserves O_1-convergence if and only if T preserves O_2-convergence.

For $f : P \to Q$, we shall compare the following properties:

(1) f preserves O_1-convergence (1′) f is $\tau_{O_1}(P) - \tau_{O_1}(Q)$ continuous
(2) f preserves O_2-convergence (2′) f is $\tau_{O_2}(P) - \tau_{O_2}(Q)$ continuous
(3) f preserves O_3-convergence (3′) f is $\tau_{O_3}(P) - \tau_{O_3}(Q)$ continuous

Theorem 10

Let P and Q be two posets and f a function from P into Q.

1. *If f preserves O_i-convergence, then f is $\tau_{O_i}(P) - \tau_{O_i}(Q)$ continuous. Moreover, (1′) and (2′) are equivalent.*
2. *If f preserves O_1-convergence, then f preserves O_2-convergence. The converse is false.*
3. *Suppose that f is isotone. Then (1), (1′), (2), (2′) are equivalent and (3) implies any of these four conditions. However none of (1), (1′), (2), (2′) implies (3).*
4. *Suppose that f is isotone, and P and Q are lattices. Then all the six conditions are equivalent.*

The notion of unbounded order convergence on Riesz spaces has received considerable attention. Let us recall that the notion of unbounded order convergence is an abstraction of almost everywhere convergence in function spaces.

Definition 11

Let P be a poset and let $F \subset P^P$. Let $i \in \{1, 2\}$.

1. The net $(x_\gamma)_{\gamma\in\Gamma}$ is said to *FO_i-converge to x in P* if $(f(x_\gamma))_{\gamma\in\Gamma}$ is O_i-convergent to $f(x)$ for every $f \in F$.
2. A subset $X \subset P$ is said to be *FO_i-closed* if there is no net in X that is FO_i-converging to a point outside of X.

1. Clearly, $x_\gamma \xrightarrow{FO_1} x \;\Rightarrow\; x_\gamma \xrightarrow{FO_2} x$, for every net $(x_\gamma)_{\gamma\in\Gamma}$, x in the poset P, and $F \subset P^P$.
2. The collection of all FO_1-closed ($=FO_2$-closed) subsets of P forms a topology on P. Denote this topology by $\tau_{FO_i}(P)$ for $i \in \{1, 2\}$.

Theorem 12

Let P be a poset and $F \subset P^P$. If a net $(x_\gamma)_{\gamma\in\Gamma}$ is FO_2-convergent to $x \in P$, then, $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that FO_1-converges to x.

A subset $X \subset P$ is FO_1-closed iff it is FO_2-closed.

Theorem 13

Let P be a poset and $F \subset P^P$. Let (Y, τ) be a topological space. For every $\varphi : P \to Y$ the following assertions are equivalent:

1. φ *is* $\tau_{FO}(P) - \tau$*-continuous;*
2. *If* $(x_\gamma)_{\gamma\in\Gamma}$ *is* FO_2*-convergent to* x *in* P*, then* $(\varphi(x_\gamma))_{\gamma\in\Gamma}$ *is convergent to* $\varphi(x)$ *w.r.t.* τ*;*
3. *If* $(x_\gamma)_{\gamma\in\Gamma}$ *is* FO_1*-convergent to* x *in* P*, then* $(\varphi(x_\gamma))_{\gamma\in\Gamma}$ *is convergent to* $\varphi(x)$ *w.r.t.* τ*.*

Unbounded order convergence can be generalized for lattices.

Proposition 14

For a lattice L the following statements are equivalent:

1. L is distributive.
2. $f_{a,b} : x \mapsto (x \wedge b) \vee a$ is a lattice homomorphism for all $a, b \in L$.
3. $g_{a,b} : x \to g_{a,b}(x) := (x \vee a) \wedge b$ is a lattice homomorphism for all $a, b \in L$.

Definition 15

In a distributive lattice L, a net $(x_\gamma)_{\gamma \in \Gamma}$ is said to be *unbounded O_i-convergent* to x ($\mathfrak{u}O_i$-convergent in short) if $x_\gamma \xrightarrow{FO_i} x$ with $F = \{f_{s,t} : s, t \in L,\ s \leq t\}$.

Theorem 16

Let $(G, +, \tau)$ be a commutative ℓ-group. Let

$$F := \{f_{s,t} : s, t \in L,\ s \leq t\}.$$

Then $x_\gamma \xrightarrow{\mathfrak{u}O_i} x$ iff $x_\gamma \xrightarrow{FO_i} x$, for every net $(x_\gamma)_{\gamma\in\Gamma}$ and x in G.

Theorem 17

Let L be a distributive lattice.

1. *If a net $(x_\gamma)_{\gamma\in\Gamma}$ is $\mathfrak{u}O_2$-convergent to $x \in L$, then, $(x_\gamma)_{\gamma\in\Gamma}$ has a subnet that $\mathfrak{u}O_1$-converges to x.*
2. $\tau_{\mathfrak{u}O_1}(L) = \tau_{\mathfrak{u}O_2}(L)$.

Corollary 18

Let τ be a topology on a distributive lattice L. Then the following conditions are equivalent:

- $x_\gamma \xrightarrow{uO_1} x \;\Rightarrow\; x_\gamma \xrightarrow{\tau} x$,
- $x_\gamma \xrightarrow{uO_2} x \;\Rightarrow\; x_\gamma \xrightarrow{\tau} x$.

Let (X, Σ, μ) be a semi-finite measure space and let L^∞ denote the Banach algebra of essentially bounded real-valued functions. We consider several useful topologies on L^∞.

1. *The norm topology* τ_∞.
2. *the topology of convergence in measure* τ_μ: Every $E \in \Sigma$ satisfying $\mu(E) < \infty$ defines an F-seminorm $\rho_E : f \mapsto \int |f| \wedge \chi_E \, \mathrm{d}\mu$ on L^∞. The Hausdorff and linear topology induced by the family $\{\rho_E \ : \ E \in \Sigma, \, \mu(E) < \infty\}$ is the topology of convergence in measure (on sets of finite measure) and is denoted by τ_μ.
3. *the strong-operator topology* σ_p*, where* $1 \leq p < \infty$.
4. *The bilinear form* $L^\infty \times L^1 \to \mathbb{R}$ *defined by* $\langle f, g \rangle \mapsto \int fg \, \mathrm{d}\mu$ *induces a duality. Amongst the locally convex topologies that are consistent with this duality, we consider the weak topology* $\sigma(L^\infty, L^1)$ *and the Mackey topology* $\tau(L^\infty, L^1)$.

Proposition 19

On L^∞ the order topology is finer than the strong operator topology, i.e $\sigma_p \subset \tau_O(L^\infty)$.

Theorem 20

1. *$\tau(L^\infty, L^1)$ is a locally convex-solid and order-continuous topology.*
2. *$\tau(L^\infty, L^1)$ is the finest Hausdorff locally convex and order-continuous topology on L^∞.*

Theorem 21

Let (X, Σ, μ) be a semi-finite measure space.

1. *For every $1 \leq p < q < \infty$*

$$\sigma(L^\infty, L^1) \subset \sigma_p \subset \sigma_q \subset \tau(L^\infty, L^1) \subset \tau_\infty, \text{ and}$$
$$\tau_\mu \subset \sigma_p,$$

   and – unless L^∞ is finite-dimensional – all of these inclusions are proper.

2. *$\tau(L^\infty, L^1) \subset \tau_O(L^\infty) \subset \tau_\infty$ and $\tau_O(L^\infty) = \tau_\infty$ if and only if L^∞ is finite-dimensional.*

3. *If (X, Σ, μ) is σ-finite, the restrictions of τ_μ and $\tau_O(L^\infty)$ to bounded parts of L^∞ are equal.*

Now we give a sufficient condition for which the topologies $\tau_O(L^\infty) \neq \tau(L^\infty, L^1)$.

For $A \in \Sigma$ let $L^\infty(A) := \{f\chi_A : f \in L^\infty\}$ and $L^1(A) := \{f\chi_A : f \in L^1\}$.

Lemma 22

Let $A \in \Sigma$.

1. *The restriction of $\sigma(L^1, L^\infty)$ to $L^1(A)$ is equal to the topology $\sigma(L^1(A), L^\infty(A))$ arising from the duality $\langle L^1(A), L^\infty(A)\rangle$.*
2. *The restriction of $\tau(L^\infty, L^1)$ to $L^\infty(A)$ is equal to the topology $\tau(L^\infty(A), L^1(A))$ arising from the duality $\langle L^1(A), L^\infty(A)\rangle$.*
3. *The order topology $\tau_O(L^\infty(A))$ is equal to the restriction of $\tau_O(L^\infty)$ to $L^\infty(A)$.*

Corollary 23

If $\tau_O(L^\infty) = \tau(L^\infty, L^1)$, then $\tau_O(L^\infty(A)) = \tau(L^\infty(A), L^1(A))$ holds for every $A \in \Sigma$

Theorem 24

Let $A \in \Sigma$ satisfy $\mu(A) \neq 0$ and such that it contains no μ-atoms. Then $\tau_O(L^\infty)$ and $\tau(L^\infty, L^1)$ are not equal.