The Choquet integral representability in a general setting

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- X is a nonempty set.
- $\mathcal{D}$ is a collection of subsets of X containing $\emptyset$.

Definition 1 (nonadditive measure)

A set function $\mu\colon \mathcal{D} \to [0,\infty]$ is called a nonadditive measure on $\mathcal{D}$ if it satisfies

1. $\mu(\emptyset) = 0$,
2. $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{D}$ and $A \subset B$.

Nonadditive measures are widely used in theory and applications and have already appeared in many papers under various names: Hausdorff dimension (Hausdorff 1918), lower/upper numerical probability (Koopman 1940), Maharam's submeasure problem (Maharam 1947), capacity (Choquet 1953/54), semivariation (Dunford-Schwartz 1955), quasimeasure (Alexiuk 1968), maxitive measure (Shilkret 1971), participation measure (Tsichritzis 1971), submeasure (Drewnowski 1972, Dobrakov 1974), fuzzy measure (Sugeno 1974), k-triangular set function (Agafanova-Klimkin 1974), game of characteristic function form, distorted measure (Aumann-Shapley 1974), belief/plausibility function (Shafer 1976), possibility measure (Zadeh 1978), pre-measure (Šipoš 1979), necessity measure (Dubois-Prade 1980), approximately additive (Kalton-Roberts 1983), decomposable measure (Weber 1984), Minkowski-Bouligrand dimension (Schroeder 1991), subjective probabilities in decision making, $\cdots\cdots$

- $\mathcal{M}_b(X)$ is the set of all finite nonadditive measures on $\mathcal{D}$.

We focus on the following continuity of nonadditive measures.

Definition 2

A nonadditive measure $\mu\colon \mathcal{D} \to [0,\infty]$ is called

- τ-inner continuous if $\mu(D_\tau) \to \mu(D)$ whenever $\{D_\tau\}_{\tau\in\Gamma}$ is a nondecreasing net, $D \in \mathcal{D}$, and $D = \bigcup_{\tau\in\Gamma} D_\tau$,
- τ-outer continuous if $\mu(D_\tau) \to \mu(D)$ whenever $\{D_\tau\}_{\tau\in\Gamma}$ is a nonincreasing net, $D \in \mathcal{D}$, and $D = \bigcap_{\tau\in\Gamma} D_\tau$,
- inner continuous if $\mu(D_n) \to \mu(D)$ whenever $\{D_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence, $D \in \mathcal{D}$, and $D = \bigcup_{n\in\mathbb{N}} D_n$,
- outer continuous if $\mu(D_n) \to \mu(D)$ whenever $\{D_n\}_{n\in\mathbb{N}}$ is a nonincreasing sequence, $D \in \mathcal{D}$, and $D = \bigcap_{n\in\mathbb{N}} D_n$,

It is obvious that

- τ-inner continuous $\Rightarrow$ inner continuous
- τ-outer continuous $\Rightarrow$ outer continuous

The Choquet integral and the asymmetric Choquet integral

The Choquet integral is a nonlinear integral and widely used in theory and its applications, such as subjective evaluation, decision making, expected utility theory, economic models under Knightian uncertainty, data mining, and others.

Definition 3 (G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953–54) 131–295.)

Let $\mu\colon \mathcal{D} \to [0,\infty]$ be a nonadditive measure and $f\colon X \to \mathbb{R}$ a $\mathcal{D}$-measurable function, that is, $\{f > t\}, \{f \geq t\} \in \mathcal{D}$ for every $t \in \mathbb{R}$.

- For $f \geq 0$, the Choquet integral of f with respect to μ is defined by

$$\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f \geq t\})dt.$$

The Choquet integral can be extended to an integral of functions taking negative values. One of them is the following:

- If μ is finite, the asymmetric Choquet integral of f is defined by

$$\mathrm{Ch}^a(\mu, f) := \int_0^\infty \mu(\{f \geq t\})dt - \int_{-\infty}^0 \{\mu(X) - \mu(\{f \geq t\})\}$$

These integrals are the honest extension of the Lebesgue integral since they are equal to the abstract Lebesgue integral if $\mathcal{D}$ is a σ-field and μ is σ-additive.

Basic properties of the asymmetric Choquet integral

The asymmetric Choquet integral has the following properties that will be important in this talk. Let $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

1. $\mathrm{Ch}^a(\mu, f) = \mathrm{Ch}(\mu, f)$ if $f \geq 0$.
2. $\mathrm{Ch}^a(\mu, f) \leq \mathrm{Ch}^a(\mu, g)$ if $f \leq g$ (monotonicity).
3. $\mathrm{Ch}^a(\mu, cf) = c\,\mathrm{Ch}^a(\mu, f)$ if $c > 0$ (positive homogeneity).
4. $\mathrm{Ch}^a(\mu, -f) = -\mathrm{Ch}^a(\bar{\mu}, f)$ (asymmetry), where $\bar{\mu}(A) := \mu(X) - \mu(X \setminus A)$.
5. $\mathrm{Ch}^a(\mu, f + c) = \mathrm{Ch}^a(\mu, f) + c\mu(X)$ for $c \in \mathbb{R}$ (translation-invariance).
6. $|\mathrm{Ch}^a(\mu, f)| \leq \mu(X)\,\|f\|_\infty$ (boundedness).
7. $\mathrm{Ch}^a(\mu, f + g) = \mathrm{Ch}^a(\mu, f) + \mathrm{Ch}^a(\mu, g)$ if f and g are comonotonic[1] ($f \sim g$ for short), i.e., $\forall x_1, x_2 \in X,\ f(x_1) < f(x_2) \Rightarrow g(x_1) \leq g(x_2)$ (comonotonic additivity).

[1] It is called similarly ordered in “Inequalities” written by Hardy, Littlewood, and Pólya.

Some characteristics of functional

We introduce the same characteristics of a functional as the asymmetric Choquet integral has.

Definition 4

Let Ψ be a vector lattice of functions $f\colon X \to \mathbb{R}$. Then a functional $I\colon \Psi \to \mathbb{R}$ is called

- monotone if $I(f) \leq I(g)$ whenever $f, g \in \Psi$ and $f \leq g$,
- positively homogeneous if $I(cf) = cI(f)$ whenever $f \in \Psi$ and $c > 0$,
- translation-invariant if $I(f + c) = I(f) + I(c)$ whenever $c \in \mathbb{R}$ and $f, c \in \Psi$,
- bounded if there is $M > 0$ such that $|I(f)| \leq M\|f\|_\infty$ for all $f \in \Psi$,
- comonotonically additive if $I(f + g) = I(f) + I(g)$ whenever $f, g \in \Psi$ and $f \sim g$.

To explain the purpose of the talk, we introduce only two results of the Choquet integral representation theorems in the literature among others.

- X is a compact Hausdorff space.
- $C_b(X)$ is the vector lattice of all (bounded) continuous functions on X.
- $I\colon C_b(X) \to \mathbb{R}$ is comonotonically additive and monotone (C.M. for short).

(1) There is a finite nonadditive measure β on 2^X such that β is outer continuous on the collection $\sum_1$ of the compact G_δ-sets and satisfies

$$I(f) = \mathrm{Ch}^a(\beta, f) \tag{1}$$

for every $f \in C_b(X)$. Hereafter, β is called a representing measure of I.

(2) Conversely, if a nonadditive measure β on 2^X is outer continuous on $\sum_1$, then the functional I defined by (1) is a C.M. functional on $C_b(X)$.

Remark 1

We can find a finite representing measure α of I that is inner continuous on the collection $\sum_2$ of the open K_σ-sets. However, it has not been discussed whether it is possible to find a representing measure that is outer continuous on $\sum_1$ and inner continuous on $\sum_2$.

- X is a locally compact Hausdorff space.
- $C_{00}(X)$ is the vector lattice of all continuous functions on X with compact support.
- $I\colon C_{00}(X) \to \mathbb{R}$ is C.M.

(1) If I is asymptotically translatable, then there is a finite nonadditive measure α on 2^X such that is α is τ-inner continuous on the collection $\mathcal{G}$ of the open sets and satisfies

$$I(f) = \mathrm{Ch}^a(\alpha, f) \tag{2}$$

for every $f \in C_{00}(X)$.

(2) Conversely, if a nonadditive measure α on 2^X is τ-inner continuous on $\mathcal{G}$, then the functional I defined by (2) is an asymptotically translatable C.M. functional on $C_{00}(X)$.

Remark 2

We can find a finite representing measure β of I that is τ-outer continuous on the collection $\mathcal{K}$ of the compact sets. However, it has not been discussed whether it is possible to find a representing measure that is τ-inner continuous on $\mathcal{G}$ and τ-outer continuous on $\mathcal{K}$.

The purpose of the study

We would like to

- obtain a representing measure that is τ-inner continuous and τ-outer continuous on the respective collections of tractable sets such as the open sets, the closed sets, and the compact sets,
- unify and improve previous Choquet integral representation theorems in the literature.

The main tools are our improvement of the Greco theorem[2] on the Choquet representation theorem in a general setting and König's scheme[3] to obtain a "continuous Daniell-Stone representation theorem".

[2] G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Sem. Mat. Univ. Padova 66 (1982) 21–42.

[3] H. König, Measure and Integration, An Advanced Course in Basic Procedures and Applications, Chapter V, Springer 1997.

The Greco theorem (1992)

We introduce the Greco theorem[4] that is our starting point in this talk. It gives a primitive form of the Choquet integral representation theorem.

- X is a nonempty set.
- Φ is a collection of functions $f\colon X \to [0,\infty]$ with $0 \in \Phi$.

Given a monotone functional $I\colon \Phi \to [0,\infty]$ with $I(0) = 0$, define the set functions $\alpha, \beta\colon 2^X \to [0,\infty]$ by

$$\alpha(A) := \sup\{I(f)\colon f \in \Phi, f \le \chi_A\}$$

and

$$\beta(A) := \inf\{I(f)\colon f \in \Phi, \chi_A \le f\}$$

for every subset A of X. Then, they are nonadditive measures on 2^X with $\alpha \le \beta$.

[4] G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Sem. Mat. Univ. Padova 66 (1982) 21–42.

In this abstract setting, Greco gave the following theorem in 1982.

Theorem 5 (Greco 1982)

Let X be a nonempty set and Φ a collection of functions $f\colon X \to [0,\infty]$ with $0 \in \Phi$. Assume that Φ satisfies the Greco condition, i.e.,

$$cf,\ f \wedge c,\ f - f \wedge c = (f-c)^+ \in \Phi$$

for every $f \in \Phi$ and $c > 0$.
If a C.M. functional $I\colon \Phi \to [0,\infty]$ with $I(0) = 0$ satisfies

- (i) $\sup_{a>0} I(f - f \wedge a) = I(f)$ *for every $f \in \Phi$ (lower marginal continuity),*
- (ii) $\sup_{b>0} I(f \wedge b) = I(f)$ *for every $f \in \Phi$ (upper marginal continuity),*

then, for any nonadditive measure λ on 2^X the following are equivalent.

- (a) $\alpha \leq \lambda \leq \beta$.
- (b) $I(f) = \mathrm{Ch}(\lambda, f)$ *for every $f \in \Phi$.*

Hence, α is the smallest representing measure of I and β is the largest one.

König's scheme (1997)

Next we recall König's scheme[5] to obtain a continuous Deniell-Stone representation, which was formed in his book in 1997.

Let X be a nonempty set. For a nonempty subset Φ of $[0,\infty]^X$, define the following collections of functions.

$$\Phi^* := \left\{ \sup_{f\in\Phi_0} f \colon \Phi_0 \text{ at most countable nonempty subsets of } \Phi \right\}$$

$$\Phi^{**} := \left\{ \sup_{f\in\Phi_0} f \colon \Phi_0 \text{ nonempty subsets of } \Phi \right\}$$

$$\Phi_* := \left\{ \inf_{f\in\Phi_0} f \colon \Phi_0 \text{ at most countable nonempty subsets of } \Phi \right\}$$

$$\Phi_{**} := \left\{ \inf_{f\in\Phi_0} f \colon \Phi_0 \text{ nonempty subsets of } \Phi \right\}$$

[5] H. König, Measure and Integration, An Advanced Course in Basic Procedures and Applications, Springer, Heidelberg, 1997.

We also define the collections of sets as follows.

$$\mathcal{H}_\Phi := \Big\{\{f > t\} \colon f \in \Phi, t > 0\Big\}$$

$$\mathcal{H}_\Phi^* := \left\{\bigcup_{H \in \mathcal{H}_0} H \colon \mathcal{H}_0 \text{ at most countable nonempty subsets of } \mathcal{H}_\Phi\right\}$$

$$\mathcal{H}_\Phi^{**} := \left\{\bigcup_{H \in \mathcal{H}_0} H \colon \mathcal{H}_0 \text{ nonempty subsets of } \mathcal{H}_\Phi\right\}$$

$$\mathcal{L}_\Phi := \Big\{\{f \geq t\} \colon f \in \Phi, t > 0\Big\}$$

$$\mathcal{L}_\Phi^* := \left\{\bigcap_{L \in \mathcal{L}_0} L \colon \mathcal{L}_0 \text{ at most countable nonempty subsets of } \mathcal{L}_\Phi\right\}$$

$$\mathcal{L}_\Phi^{**} := \left\{\bigcap_{L \in \mathcal{H}_0} L \colon \mathcal{L}_0 \text{ nonempty subsets of } \mathcal{L}_\Phi\right\}$$

The regulalizations of α and β

Recall that

$$\alpha(A) := \sup\{I(f) \colon f \in \Phi, f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) \colon f \in \Phi, \chi_A \leq f\}.$$

By using the collections of sets on the previous slide, define the regularizations of α and β by

$$\alpha^\bullet(A) := \inf\{\alpha(H) \colon A \subset H, H \in \mathcal{H}_\Phi\},$$
$$\beta^\bullet(A) := \sup\{\beta(L) \colon L \subset A, L \in \mathcal{L}_\Phi\},$$
$$\alpha^*(A) := \inf\left\{\alpha(H^*) \colon A \subset H^*, H^* \in \mathcal{H}_\Phi^*\right\},$$
$$\beta^*(A) := \sup\left\{\beta(L^*) \colon L^* \subset A, L^* \in \mathcal{L}_\Phi^*\right\},$$
$$\alpha^{**}(A) := \inf\left\{\alpha(H^{**}) \colon A \subset H^{**}, H^{**} \in \mathcal{H}_\Phi^{**}\right\},$$
$$\beta^{**}(A) := \sup\left\{\beta(L^{**}) \colon L^{**} \subset A, L^{**} \in \mathcal{L}_\Phi^{**}\right\}.$$

for every subset A of X, which are nonadditive measures.

An improvement of the Greco theorem

Theorem 6 (K: 2022)

Let Φ be a lattice of bounded functions $f\colon X \to [0,\infty)$ with $0 \in \Phi$. Assume that

- (i) *Φ satisfies the Greco condition, i.e., $cf, f \wedge c, f - f \wedge c = (f-c)^+ \in \Phi$ for every $f \in \Phi$ and $c > 0$.*
- (ii) *Φ separates sets in $\mathcal{L}_\Phi^{**}$ and $\mathcal{H}_\Phi^{**}$, i.e., for any $L^{**} \in \mathcal{L}_\Phi^{**}$ and $H^{**} \in \mathcal{H}_\Phi^{**}$ with $L^{**} \subset H^{**}$, there is an $f \in \Phi$ such that $\chi_{L^{**}} \le f \le \chi_{H^{**}}$.*

Let $I\colon \Phi \to [0,\infty)$ be a C.M. functional satisfying

- (iii) $\inf_{a>0} I(f \wedge a) = 0$ *for every $f \in \Phi$.*

Then the following hold.

1. *For any nonadditive measure μ on 2^X, the following are equivalent.*
   - (a) $\alpha \le \lambda \le \beta$.
   - (b) $I(f) = \mathrm{Ch}(\lambda, f)$ *for every $f \in \Phi$.*
2. *The regularizations of α and β are compatible, that is,*

$$\alpha \le \beta^\bullet \le \beta^* \le \beta^{**} \le \alpha^{**} \le \alpha^* \le \alpha^\bullet \le \beta.$$

Theorem 6 (Continued)

3. *For any* $L^{**} \in \mathcal{L}_\Phi^{**}$*, we have*
$$\alpha^\bullet(L^{**}) = \alpha^*(L^{**}) = \alpha^{**}(L^{**}) = \beta(L^{**}) = \beta^{**}(L^{**}) < \infty.$$
4. *For any* $H^{**} \in \mathcal{H}_\Phi^{**}$*, we have*
$$\alpha^{**}(H^{**}) = \alpha(H^{**}) = \beta^\bullet(H^{**}) = \beta^*(H^{**}) = \beta^{**}(H^{**}).$$
5. *Assume that*
   - $\mathcal{L}_\Phi^{**}$ *is a compact system, i.e., every nonempty subcollection of* $\mathcal{L}_\Phi^{**}$ *whose intersection is empty has a further subcollectoin whose intersection is empty,*
   - $(1 - f) \wedge g \in \Phi_{**}$ *whenever* $f, g \in \Phi$ *and* $0 \leq f \leq 1$*,*

   then α^{**} *is* τ*-inner continuous on* $\mathcal{H}_\Phi^{**}$*, and* β^{**} *is* τ*-outer continuous on* $\mathcal{L}_\Phi^{**}$*.*

By Theorem 6 (the improved Greco theorem) we see that if $\mathcal{L}_\Phi^{**}$ is a compact system and $(1 - f) \wedge g \in \Phi_{**}$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$, then α^{**} is a representing measure we are looking for since

- α^{**} is τ-inner continuous on $\mathcal{H}_\Phi^{**}$ by (5),
- $\alpha^{**} = \beta^{**}$ on $\mathcal{L}_\Phi^{**}$ by (3), and β^{**} is τ-outer continuous on $\mathcal{L}_\Phi^{**}$ by (5).

Likewise, we see that β^{**} is another representing measure we are looking for.

Note that our improvement of the Greco theorem (Theorem 6) can be only applied to a functional defined on the collection Ψ of nonnegative functions.

To handle the case where Ψ contains functions taking negative values and does not have the identity, say, $\Psi = C_{00}(X)$ and $C_0(X)$, the following notion of a functional is needed to obtain a representing measure.

Definition 7

Let Ψ be a vector lattice of functions $f\colon X \to \mathbb{R}$. A functional $I\colon \Psi \to \mathbb{R}$ is called asymptotically translatable if for any $f \in \Psi$, any nondecreasing net $\{g_\tau\}_{\tau\in\Gamma} \subset \Psi$, and any $c > 0$, if $0 \le g_\tau \le c$ and $f + g_\tau \ge 0$ for all $\tau \in \Gamma$ and $g_\tau \uparrow c$, then

$$\lim_{\tau\in\Gamma} I(f + g_\tau) = I(f) + \lim_{\tau\in\Gamma} I(g_\tau).$$

Remark 3

The asymmetric Choquet integral $\mathrm{Ch}^a(\mu, \cdot)$ is asymptotically translatable if μ is τ-inner continuous on $\mathcal{H}_\Phi^{**}$ and $\Phi = \Psi^+$.

The continuous Choquet integral representation theorem

Theorem 8 (K: 2022)

Let Ψ be a vector lattice of bounded functions $f\colon X \to \mathbb{R}$. Let $\Phi = \Psi^+$ satisfy

- (i) *Φ is Stonean, i.e., $f \wedge 1 \in \Phi$ for every $f \in \Phi$,*
- (ii) *Φ separates sets in $\mathcal{L}_\Phi^{**}$ and $\mathcal{H}_\Phi^{**}$,*
- (iii) *Φ possesses an approximate identity net,*
- (iv) *$\mathcal{L}_\Phi^{**}$ is a compact system,*
- (v) *$(1-f) \wedge g \in \Phi_{**}$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$.*

Let $I\colon \Psi \to \mathbb{R}$ be an asymptotically translatable C.M. functional satisfying $\inf_{a>0} I(f \wedge a) = 0$ for every $f \in \Phi$. Then there is a $\mu \in \mathcal{M}_b(X)$ such that

- (a) *$I(f) = \mathrm{Ch}^a(\mu, f)$ for every $f \in \Psi$.*
- (b) *μ is τ-inner continuous on $\mathcal{H}_\Phi^{**}$ and τ-outer continuous on $\mathcal{L}_\Phi^{**}$.*

In addition, the following hold.

1. *If $\mu \in \mathcal{M}_b(X)$ satisfies (a) and (b), then it is uniquely determined on $\mathcal{H}_\Phi^{**} \cup \mathcal{L}_\Phi^{**}$.*
2. *For any $\mu \in \mathcal{M}_b(X)$ satisfying (b), if the functional I on Ψ is defined by (a), then it is an asymptotically translatable C.M. functional on Ψ satisfying $\inf_{a>0} I(f \wedge a) = 0$ for every $f \in \Phi$.*

A variety of the continuous Choquet integral representation theorems

We introduce various continuous Choquet integral representation theorems that can be derived from Theorems 6 and 8 in a general setting.

1. X is locally compact and $I\colon C_{00}^{+}(X) \to \mathbb{R}$ is C.M. $\Rightarrow \mu$ is τ-inner continuous on the open sets and τ-outer continuous on the compact sets (an improvement of Narukawa in 2007).
2. X is locally compact and $I\colon C_{0}^{+}(X) \to \mathbb{R}$ is bounded C.M. $\Rightarrow \mu$ is τ-inner continuous on the open sets and τ-outer continuous on the compact sets (an improvement of K. in 2011).
3. X is compact and $I\colon C_b(X) \to \mathbb{R}$ is C.M. $\Rightarrow \mu$ is τ-inner continuous on the open sets and τ-outer continuous on the compact sets since every C.M. functional is asymptotically translatable by Dini's theorem (an improvement of Zhou's theorem in 1998).
4. X is locally compact and $I\colon C_{00}(X) \to \mathbb{R}$ is asymptotically translatable C.M. $\Rightarrow \mu$ is τ-inner continuous on the open sets and τ-outer continuous on the compact sets (an improvement of K. in 2013).
5. X is locally compact and $I\colon C_0(X) \to \mathbb{R}$ is asymptotically translatable, bounded C.M. $\Rightarrow \mu$ is τ-inner continuous on the open sets and τ-outer continuous on the compact sets (an improvement of K. in 2013).

Thank you so much for your kind attention!