Nuclear operators and operator-valued Borel measures

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- For X being a compact Hausdorff space, nuclear operators $T : C(X) \to F$ were studied by L. Schwartz [S] and Tong [T]. In particular, *the Bochner-type representation* of nuclear operators was derived.

Theorem [T, Proposition 1.2]

A bounded operator $T : C(X) \to F$ is nuclear if and only if it has a Bochner kernel, i.e., there exist a finite real valued measure μ on X and a Bochner integrable function $f : X \to F$ such that

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[Po] Popa, D., *Nuclear operators on* $C(T, X)$, Stud. Cerc. Mat., 42 no. 1 (1990), 47–50.

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Nuclear operators

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Let E' and F' denote the Banach duals of E and F, respectively. By $\mathcal{L}(E, F)$ we denote the space of all bounded linear operators from E to F, equipped with the norm $\|\cdot\|$ of the uniform operator topology.

An operator $U \in \mathcal{L}(E, F)$ is **nuclear** if there exist sequences (x_n') in E' and (y_n) in F such that

$$U(x) = \sum_{n=1}^{\infty} x_n'(x) y_n \quad \text{for} \quad x \in E,$$

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Nuclear operators

The **nuclear norm** of a nuclear operator $U : E \to F$ is defined by

$$\|U\|_{nuc} := \inf \sum_{n=1}^{\infty} \|x_n'\|_{E'} \cdot \|y_n\|_F,$$

where the infimum is taken over all sequences (x_n') in E' and (y_n) in F such that $U(x) = \sum_{n=1}^{\infty} x_n'(x) y_n$ for $x \in E$ and $\sum_{n=1}^{\infty} \|x_n'\|_{E'} \|y_n\|_F < \infty$.

The space $\mathcal{N}(E, F)$ of all nuclear operators $U : E \to F$, equipped with the nuclear norm $\|\cdot\|_{nuc}$ is a Banach space (see [P, Proposition, p. 51]).

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Strict topology β

Let $(X, \mathcal{T})$ be a completely regular Hausdorff space. Let $C_b(X, E)$ denote the space of all E-valued bounded continuous functions defined on X.

Definition 1.1. [B], [F]

The strict topology β on $C_b(X, E)$ is generated by the family of the seminorms:

$$p_v(f) := \sup_{t \in X} v(t) \|f(t)\|_E \quad \text{for} \quad f \in C_b(X, E),$$

where $v : X \to [0, \infty)$ is a bounded function such that the set $\{t \in X : v(t) \geq \varepsilon\}$ is compact for every $\varepsilon > 0$.

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Strict topology β

Theorem 1.1. [K]

- $\tau_c \subset \beta \subset \tau_u$ and if X is a compact Hausdorff space, then $\beta = \tau_u$.
- β and τ_u have the same bounded subsets.
- β is the finest locally convex Hausdorff topology agreeing with topology τ_c on τ_u-bounded subsets of $C_b(X, E)$.

[K] Khan, L.A., *The strict topology on a space of vector-valued functions*, Proc. Edinburgh Math. Soc., 22 no. 1 (1979), 35–41.

Basic concepts of measure theory

By $\mathcal{B}o$ we denote the σ-algebra of Borel sets in X.
Let $M(X)$ denote the space of all countably additive regular scalar Borel measures.

The variation $|\mu|$ of the measure $\mu : \mathcal{B}o \to E$ on $A \in \mathcal{B}o$ is defined by:

$$|\mu|(A) := \sup \sum \|\mu(A_i)\|_E,$$

where the supremum is taken over all finite $\mathcal{B}o$-partitions (A_i) of A (see [DU, Definition 4, p. 2]).

Let $M(X, E')$ denote the space of all countably additive measures $\mu : \mathcal{B}o \to E'$ of bounded variation ($|\mu|(X) < \infty$) and such that for each $x \in E$, $\mu_x \in M(X)$, where $\mu_x(A) := \mu(A)(x)$ for $A \in \mathcal{B}o$.

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The semivariation $\widetilde{m}$ of $m : \mathcal{B}o \to \mathcal{L}(E, F)$ on $A \in \mathcal{B}o$ is defined by

$$\widetilde{m}(A) := \sup \left\| \sum m(A_i)(x_i) \right\|_F ,$$

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By $M(X, \mathcal{L}(E, F))$ we denote the space of all measures $m : \mathcal{B}o \to \mathcal{L}(E, F)$ such that $\widetilde{m}(X) < \infty$ and for each $y' \in F'$, $m_{y'} \in M(X, E')$, where

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Riemann–Stieltjes integral

Definition 1.2. [F], [Go], [N, Definition 2.2])

Let $m \in M(X, \mathcal{L}(E,F))$. We say that $f \in C_b(X,E)$ is ***m*-integrable in the Riemann–Stieltjes sense** over $A \in \mathcal{B}o$ provided there exists $y_A \in F$ such that given $\varepsilon > 0$ there exists a finite $\mathcal{B}o$-partition $\mathcal{P}_\varepsilon$ of A such that

$$\left\| y_A - \sum_{i=1}^{n} m(A_i)(f(t_i)) \right\|_F \leq \varepsilon$$

if $\{A_1, \ldots, A_n\}$ is any $\mathcal{B}o$-partition of A refining $\mathcal{P}_\varepsilon$ and $\{t_1, \ldots, t_n\}$ is any choice of points of A such that $t_i \in A_i$ for $i = 1, \ldots, n$.

Then $\int_A f(t)\, dm := y_A$ will be called a **Riemann–Stieltjes integral** of f with respect to m over $A \in \mathcal{B}o$.

[Go] Goodrich, R.K., *A Riesz representation theorem*, Proc. Amer. Math. Soc., 24 (1970), 629–636.

[N] Nowak, M., *A Riesz representation theory for completely regular Hausdorff spaces and its applications*, Open Math., 14 (2016), 474–496.

Riesz representation theorem for functionals

Theorem 1.2. [Gi], [N, Theorem 2.4]

For a linear functional $\Phi : C_b(X,E) \to \mathbb{R}$ the following statements are equivalent:

(i) Φ is β-continuous.

(ii) There exists a unique $\mu \in M(X,E')$ such that

$$\Phi(f) = \Phi_\mu(f) = \int_X f(t)\, d\mu \quad \text{for all} \quad f \in C_b(X,E).$$

Moreover, $\|\Phi_\mu\| = |\mu|(X)$.

By $C_b(X,E)'_\beta$ we will denote the topological dual of $(C_b(X,E),\beta)$. Hence $C_b(X,E)'_\beta$ can be identified with $M(X,E')$.

[Gi] Giles, R., *A generalization of the strict topology*, Trans. Amer. Math. Soc., 161 (1971), 467–474.

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Riesz representation theorem for operators

Let $i_F : F \to F''$ denote the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F.

Theorem 1.3. [N, Theorem 3.2]

Assume that $T : C_b(X, E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous linear operator. Then there exists a unique measure $m : \mathcal{B}o \to \mathcal{L}(E, F'')$ (called the *representing measure of* T) such that the following statements hold:

(i) m is tight, i.e., for all $\varepsilon > 0$ there exists a compact set K such that $\widetilde{m}(X \setminus K) \leq \varepsilon$.

(ii) The mapping $F' \ni y' \mapsto m_{y'} \in M(X, E')$ is $(\sigma(F', F), \sigma(M(X, E'), C_b(X, E)))$-continuous.

(iii) Every $f \in C_b(X, E)$ is m-integrable in the Riemann–Stieltjes sense over X .

(iv) For $f \in C_b(X, E)$, $\int_X f\, dm \in i_F(F)$ and $T(f) = j_F\left(\int_X f\, dm\right)$.

(v) For each $y' \in F'$, $y'(T(f)) = \int_X f(t)\, dm_{y'}$ for $f \in C_b(X, E)$.

Conversely, assume that a measure $m : \mathcal{B}o \to \mathcal{L}(E, F'')$ satisfies the conditions (i) and (ii). Then there exists a unique $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X, E) \to F$ such that (iii), (iv) and (v) hold. Moreover: $\|T\| = \widetilde{m}(X)$.

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Riesz representation theorem for operators

Let $i_F : F \to F''$ denote the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F.

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Strongly bounded operators

A $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is said to be **strongly bounded** if its representing measure m has the strongly bounded semivariation $\widetilde{m}$, i.e., $\widetilde{m}(A_n) \to 0$ whenever (A_n) is a pairwise disjoint sequence in $\mathcal{B}o$.

Note that if a linear operator $T : C_b(X,E) \to F$ is $(\beta, \|\cdot\|_F)$-weakly compact, then T is $(\beta, \|\cdot\|_F)$-continuous and strongly bounded (see [N, Theorem 6.2]).

It is known that if a $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is strongly bounded and m is its representing measure, then $m(A)(x) \in i_F(F)$ for $A \in \mathcal{B}o$, $x \in E$ (see [N, §4]). Then one can define a measure $m_F : \mathcal{B}o \to \mathcal{L}(E,F)$ by

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A completely regular Hausdorff space X is said to be a k-**space** if each set which meets every compact subset in a closed set must be closed.

From now on we will assume that $(X, \mathcal{T})$ is a k-space.

Theorem 1.4. [N, Theorem 4.3 and Corollary 4.4]

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Based on the paper:

[NS] Nowak, M., J. Stochmal, *Nuclear operators on $C_b(X, E)$ and the strict topology*, Math. Slovaca, 68 no. 1 (2018), 135–146.

[St] Stochmal, J., *A characterization of nuclear operators on spaces of vector-valued continuous functions with the strict topology*, submitted.

Nuclear operators on $C_b(X, E)$

Definition 2.1. [Sch, Chap. 3, §7, 7.1]

A linear operator $T : C_b(X, E) \to F$ is said to be **nuclear** between the locally convex space $(C_b(X, E), \beta)$ and Banach space $(F, \| \cdot \|_F)$ if there exist an β-equicontinuous sequence (Φ_n) in $C_b(X, E)'_\beta$, a bounded sequence (y_n) in F and $(\lambda_n) \in \ell^1$ such that

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \Phi_n(f) y_n \quad \text{for} \quad f \in C_b(X, E). \tag{2.1}$$

Then T is $(\beta, \| \cdot \|_F)$-compact (see [Sch, Chap. 3, §7, Corollary 1]). Moreover, let us put

$$\|T\|_{\beta\text{-}nuc} := \inf \left\{ \sum_{n=1}^{\infty} |\lambda_n| \|\Phi_n\| \|y_n\|_F \right\},$$

where the infimum is taken over all sequences (Φ_n) in $C_b(X, E)'_\beta$, (y_n) in F and $(\lambda_n) \in \ell^1$ such that T admits a representation (2.1).

[Sch] Schaeffer, H.H., Topological Vector Spaces, Springer-Verlag, 1971.

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Nuclear operators on $C_b(X, E)$

Theorem 2.1. [KhC, Lemma 2]

For a subset $\mathcal{M}$ of $M(X, E')$ the following statements are equivalent:

(i) $\{\Phi_\mu : \mu \in \mathcal{M}\}$ is a β-equicontinuous subset of $C_b(X, E)'_\beta$.

(ii) $\sup_{\mu\in\mathcal{M}} |\mu|(X) < \infty$ and $\mathcal{M}$ is uniformly tight, that is, for every $\varepsilon > 0$, there exists a compact subset K of X such that $\sup_{\mu\in\mathcal{M}} |\mu|(X \setminus K) \leq \varepsilon$.

Proposition 2.2.

A linear operator $T : C_b(X, E) \to F$ is nuclear if and only if there exist a bounded and uniformly tight sequence (μ_n) in $M(X, E')$, a bounded sequence (y_n) in F and $(\lambda_n) \in \ell^1$ such that

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \left(\int_X f(t)\, d\mu_n \right) y_n \quad \text{for} \quad f \in C_b(X, E).$$

[KC] Khurana, S.S., S.A. Choo, *Strict topology and P-spaces*, Proc. Amer. Math. Soc., 61 (1976), 280–284.

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Proposition 2.2.

A linear operator $T : C_b(X, E) \to F$ is nuclear if and only if there exist a bounded and uniformly tight sequence (μ_n) in $M(X, E')$, a bounded sequence (y_n) in F and $(\lambda_n) \in \ell^1$ such that

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \left(\int_X f(t)\, d\mu_n \right) y_n \quad \text{for} \quad f \in C_b(X, E).$$

[KC] Khurana, S.S., S.A. Choo, *Strict topology and P-spaces*, Proc. Amer. Math. Soc., 61 (1976), 280–284.

Nuclear operators on $C_b(X,E)$

Theorem 2.3. [NS, Theorem 3.1]

Assume that $T : C_b(X,E) \to F$ is a nuclear operator and m is its representing measure. Then the following statements hold:

(i) For each $A \in \mathcal{B}o$, $m_F(A) \in \mathcal{N}(E,F)$.

(ii) $m_F : \mathcal{B}o \to \mathcal{N}(E,F)$ is $\|\cdot\|_{nuc}$-countably additive.

(iii) $|m_F|_{nuc} \in M^+(X)$, where $|m_F|_{nuc}$ stands for the variation of m_F with respect to the norm $\|\cdot\|_{nuc}$ in $\mathcal{N}(E,F)$.

(iv) If E' has the Radon–Nikodym Property, then there exists a function $H \in L^1(|m_F|_{nuc}, \mathcal{N}(E,F))$ such that

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Let $T : C_b(X, E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous strongly bounded linear operator. If

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then T is a nuclear operator between the locally convex space $(C_b(X, E), \beta)$ and the Banach space F and $\|T\|_{\beta\text{-}nuc} = |m_F|_{nuc}(X)$.

Nuclear operators on $C_b(X, E)$

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Nuclear operators on $C_b(X, E)$

Sketch of proof.

Step 1. Using the Lebesgue dominated convergence theorem, Theorem 1.4 and [N, Theorem 5.2], we get that

$$T(f) = \int_X H(t)(f(t))\, d|m_F|_{nuc} \quad \text{for} \quad f \in C_b(X, E).$$

Step 2. Let $L^1(|m_F|_{nuc})\hat{\otimes}_\pi \mathcal{N}(E, F)$ denote the projective tensor product of the Banach spaces $L^1(|m_F|_{nuc})$ and $\mathcal{N}(E, F)$, equipped with the completed norm π (see [DU, p. 227]).

Note that for $W \in L^1(|m_F|_{nuc})\hat{\otimes}_\pi \mathcal{N}(E, F)$ we have

$$\pi(W) := \inf \left\{ \sum_{n=1}^{\infty} |\lambda_n| \|v_n\|_1 \|U_n\|_{nuc} \right\},$$

where the infimum is taken over all sequences (v_n) in $L^1(|m_F|_{nuc})$ and (U_n) in $\mathcal{N}(E, F)$ with $\lim_n \|v_n\|_1 = 0 = \lim_n \|U_n\|_{nuc}$ and $(\lambda_n) \in \ell^1$ such that $W = \sum_{n=1}^{\infty} \lambda_n (v_n \otimes U_n)$ is in the π-norm (see [R, Proposition 2.8, pp. 21–22]).

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Nuclear operators on $C_b(X, E)$

It is known that $L^1(|m_F|_{nuc})\hat{\otimes}_\pi \mathcal{N}(E,F)$ is isometrically isomorphic to the Banach space $(L^1(|m_F|_{nuc}, \mathcal{N}(E,F)), \|\cdot\|_1)$ throughout the isometry J, where

$$J(v \otimes U) := v(\cdot)U \quad \text{for} \quad v \in L^1(|m_F|_{nuc}),\ U \in \mathcal{N}(E,F)$$

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Then there exist (v_n) in $L^1(|m_F|_{nuc})$ and (U_n) in $\mathcal{N}(E,F)$ with $\lim_n \|v_n\|_1 = 0 = \lim_n \|U_n\|_{nuc}$ and $(\alpha_n) \in \ell^1$ so that

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Hence we have

$$H = J\left(\sum_{n=1}^{\infty} \alpha_n v_n \otimes U_n\right) = \sum_{n=1}^{\infty} \alpha_n v_n(\cdot) U_n \quad \text{in} \quad L^1(|m_F|_{nuc}, \mathcal{N}(E,F)).$$

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$$T(f) = \sum_{n=1}^{\infty} \alpha_n \int_X v_n(t) U_n(f(t))\, d|m_F|_{nuc} \quad \text{for} \quad f \in C_b(X,E).$$

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Nuclear operators on $C_b(X,E)$

For $n \in \mathbb{N}$, we can choose sequences $(x'_{n,k})$ in E' and $(y_{n,k})$ in F so that

$$U_n(x) = \sum_{k=1}^{\infty} x'_{n,k}(x) y_{n,k} \quad \text{for} \quad x \in E.$$

Then for every $f \in C_b(X,E)$, we have

$$\begin{aligned} T(f) &= \sum_{n=1}^{\infty} \alpha_n \int_X \left(v_n(t) \sum_{k=1}^{\infty} x'_{n,k}(f(t)) y_{n,k} \right) d|m_F|_{nuc} \\ &= \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{\infty} \|x'_{n,k}\|_{E'} \|y_{n,k}\|_F \left(\int_X v_n(t) \frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}} \, d|m_F|_{nuc} \right) \frac{y_{n,k}}{\|y_{n,k}\|_F}. \end{aligned}$$

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$$= \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{\infty} \|x'_{n,k}\|_{E'} \|y_{n,k}\|_F \left(\int_X v_n(t) \frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}} \, d|m_F|_{nuc} \right) \frac{y_{n,k}}{\|y_{n,k}\|_F}.$$

Step 3. For $n, k \in \mathbb{N}$, denote

$$\Phi_{n,k}(f) := \int_X v_n(t) \frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}} \, d|m_F|_{nuc} \quad \text{for} \quad f \in C_b(X, E).$$

Now, we should show that $(\Phi_{n,k})$ is β-equicontinuous sequence in $C_b(X, E)'_\beta$. In view of [F, Theorem 3.2] it is enough to prove that $(\Phi_{n,k}|_{B_1})$ is τ_c-equicontinuous sequence at 0, where $B_1 = \{f \in C_b(X, E) : \|f\| \leq 1\}$.

Step 4. To complete the proof we need to show $\|T\|_{\beta\text{-}nuc} = |m_F|_{nuc}(X)$.

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Corollary 2.5. [St]

Assume that $T : C_b(X, E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous strongly bounded linear operator and E' has the Radon–Nikodym property. Then the following statements are equivalent:

(i) T is a nuclear operator between the locally convex space $(C_b(X, E), \beta)$ and the Banach space F.

(ii) $m_F(A) \in \mathcal{N}(E, F)$ for $A \in \mathcal{B}o$ and $|m_F|_{nuc} \in M^+(X)$ and there exists $H \in L^1(|m_F|_{nuc}, \mathcal{N}(E, F))$ such that

$$m_F(A) = \int_A H(t)\, d|m_F|_{nuc} \quad \text{for} \quad A \in \mathcal{B}o.$$

In this case: $\|T\|_{\beta\text{-}nuc} = |m_F|_{nuc}(X)$.

Nuclearity of conjugate operator

Let $T : C_b(X, E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous linear operator and m be its representing measure. Then its conjugate operator $T' : F' \to M(X, E')$ is given by

$$T'(y') := m_{y'} \quad \text{for} \quad y' \in F'.$$

Corollary 2.6. [St]

Assume that $T : C_b(X, E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous linear operator. If E' has the Radon–Nikodym property and F is reflexive, then the following statements are equivalent:

(i) T is a nuclear operator between the locally convex space $(C_b(X, E), \beta)$ and the Banach space F.

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Assume that $T : C_b(X, E) \to F$ is a $(\beta, \| \cdot \|_F)$-continuous linear operator. If E' has the Radon–Nikodym property and F is reflexive, then the following statements are equivalent:

(i) T is a nuclear operator between the locally convex space $(C_b(X, E), \beta)$ and the Banach space F.

(ii) T' is nuclear.

In this case: $\|T'\|_{nuc} = \|T\|_{\beta\text{-}nuc}$.

Nuclearity of conjugate operator

Let $T : C_b(X, E) \to F$ be a $(\beta, \| \cdot \|_F)$-continuous linear operator and m be its representing measure. Then its conjugate operator $T' : F' \to M(X, E')$ is given by

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Thank you for your attention!