Essential norms of pointwise multipliers between distinct Köthe spaces

Positivity XI

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Contents

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- Schep's response

A. R. Schep, *The essential spectrum, norm, and spectral radius of abstract multiplication operators*, Concrete Operators 10 (2023).

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T. Kiwerski and J. Tomaszewski, *Essential norms of pointwise multipliers acting between Köthe spaces: The non-algebraic case*, preprint available on arXiv.org.

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$$\begin{aligned} \|T\colon X \to Y\|_e &:= \|T\colon X \to Y\|_{\mathscr{L}(X,Y)/\mathscr{K}(X,Y)} \\ &= \operatorname{dist}_{\mathscr{L}(X,Y)}(T\colon X \to Y, \mathscr{K}(X, Y)) \\ &= \inf\{\|T - K\|_{X\to Y} : K \in \mathscr{K}(X, Y)\}. \end{aligned}$$

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Every symbol $\lambda \in M(X, Y)$ induces **multiplication operator** $M_\lambda\colon X \to Y$ given by

$$M_\lambda x := \lambda x \text{ for } x \in X$$

Some examples of pointwise multipliers

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- $M(L_M, L_N) = L_{N \ominus M}$
- $M(X, X) \equiv L_\infty(\mu)$ for any Köthe space X.

Contents

Definition: Order continuity

Let X be a Banach function space. We say that $f \in X$ is an **order continuous** element if

$$\|f\chi_{A_n}\|_X \to 0$$

for any sequence (A_n) satisfying $A_n \downarrow \emptyset$, that is, $\chi_{A_n} \downarrow 0$ as $n \to \infty$.

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- $(L_p(\mu))_o = L_p(\mu)$ for $1 \leqslant p < \infty$,
- $(L_\infty)_o = \{0\}$, but $(\ell_\infty)_o = c_0$.

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$$\|M_\lambda \colon X \to Y\|_e = \|\lambda\|_{M(X,Y)} . \tag{1}$$

In particular, there are no non-trivial compact multiplication operators between X and Y.

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$$\|M_\lambda \colon X \to Y\|_e = \lim_{n\to\infty} \left\| \lambda \chi_{\{n,n+1,\ldots\}} \right\|_{M(X,Y)} .$$

Corollary

Let X be an order continuous Köthe sequence space. Then

$$\|M_\lambda \colon X \circlearrowleft\|_e = \lim_{n\to\infty} \left\|\lambda\chi_{\{n,n+1,\ldots\}}\right\|_{\ell_\infty} = \lim_{n\to\infty}\left(\sup_{m\geqslant n}|\lambda_m|\right) = \limsup_{n\to\infty}|\lambda_n|.$$

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Let X and Y be two Köthe sequence spaces. Suppose that either the space X is reflexive or the space Y is order continuous. Then

$$\operatorname{dist}_{\mathscr{L}(X,Y)}(M_\lambda\colon X \to Y, \mathscr{K}(X,Y)) = \operatorname{dist}_{M(X,Y)}(\lambda, M(X,Y)_o).$$

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$$\mathrm{dist}_{\mathscr{L}(X,Y)}(M_\lambda \colon X \to Y, \mathscr{K}(X,Y)) = \mathrm{dist}_{M(X,Y)}(\lambda, M(X,Y)_o).$$

In particular, the multiplication operator $M_\lambda \colon X \to Y$ is compact if, and only if, $\lambda \in M(X,Y)_o$.

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$$\|M_\lambda \colon X \to Y\|_e \approx \max\left\{\|M_\lambda \colon X|_{\Omega_c} \to Y|_{\Omega_c}\|_e, \|M_\lambda \colon X|_{\Omega_a} \to Y|_{\Omega_a}\|_e\right\}$$

with an equivalence involving universal constants only.

Contents

Pitt's theorem

Every operator from ℓ_p into ℓ_q is compact if, and only if, $1 \leqslant q < p \leqslant \infty$ (with the convention that whenever $p = \infty$, then we are working with c_0 instead of ℓ_∞), or, pictographically,

$$\mathscr{L}(\ell_p, \ell_q) = \mathscr{K}(\ell_p, \ell_q)$$

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Theorem: Pitt's theorem for pointwise multipliers

Let X and Y be two Köthe sequence spaces. Then every multiplication operator M_λ acting from X into Y is compact if, and only if, the space $M(X, Y)$ is order continuous.

Definition: Spaces of analytic functions

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Let $\mathscr{H}(\mathbb{D})$ be the space of all analytic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. With a linear subspace, say $H(\mathbb{D})$, of $\mathscr{H}(\mathbb{D})$ we can associate the space

$$\widehat{H}(\mathbb{D}) := \left\{ \left\{ \widehat{f}(n) \right\}_{n=0}^{\infty} : \sum_{n=0}^{\infty} \widehat{f}(n) \chi_n \in H(\mathbb{D}) \right\}$$

of Taylor's coefficients of functions from $H(\mathbb{D})$, where $\chi_n(z) = z^n$ for $z \in \mathbb{D}$ and $n = 0, 1, 2, \ldots$.

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Definition: Fourier multipliers

Let $H(\mathbb{D})$ be a Banach space of analytic function on the unit disc and Y be a Köthe sequence space. For $\lambda \in M(\widehat{H}(\mathbb{D}), Y)$ we define the **Fourier multiplier** $\mathscr{M}_\lambda \colon H(\mathbb{D}) \to Y$ as

$$\mathscr{M}_\lambda \colon f \mapsto \left\{\lambda_n \widehat{f}(n)\right\}_{n=0}^{\infty},$$

where $f = \sum_{n=0}^{\infty} \widehat{f}(n)\chi_n \in H(\mathbb{D})$.

Theorem: Essential norm of Fourier multiplier

Let $H(\mathbb{D})$ be a Banach space of analytic function on the unit disc intermediate between $H_\infty(\mathbb{D})$ and $H_2(\mathbb{D})$, that is, $H_\infty \hookrightarrow H \hookrightarrow H_2$.

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In particular, the Fourier multiplier $\mathscr{M}_\lambda \colon H(\mathbb{D}) \to Y$ is compact if and only if $\lambda \in M(\ell_2, Y)_o$.

Thank you for attention!