Fubini's theorem for a vector-valued Daniell integral

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Introduction.

Let T be an arbitrary set and let $\mathfrak{E}^T$ be the set of all functions defined on T with values in the universally complete Riesz space $\mathfrak{E}$ with weak order unit E. We note that $\mathfrak{E}$ is an f-algebra with E as algebraic unit.
By defining the operations point wise, i.e., for all $t \in T$,

$$(X+Y)(t) := X(t)+Y(t), \ \ (cX)(t) := cX(t), \ \ X \leq Y \iff X(t) \leq Y(t),$$

it follows that $\mathfrak{E}^T$ is a Dedekind complete Riesz space with weak order unit $(E(t))$, where $E(t) = E$ for all $t \in T$.

The Daniell integral

Let $\mathbb{L}$ be a Riesz subspace of $\mathfrak{E}^T$ contained in the ideal generated in $\mathfrak{E}^T$ by the weak order unit $E = E(t)$ for all $t \in T$.

Definition

A positive linear operator $I : \mathbb{L} \to \mathfrak{E}$ is called a *vector-valued I-integral* whenever, for every sequence (X_n) in $\mathbb{L}$ that satisfies $X_n(t) \downarrow 0$ for every $t \in \mathfrak{E}^T$, it follows that $I(X_n) \downarrow 0$.

We refer to $\mathbb{L}$ as the *initial domain* of the integral I.
The integral is a positive σ-order continuous linear operator mapping $\mathbb{L}$ into $\mathfrak{E}$.
The vector-valued I-integral is then extended by the well-known Daniell extension process to a positive integral. We denote the extended integral again by I. A function $X(t)$ is called I-summable if $I(X(t)) \in \mathfrak{E}$. The set of all I-summable functions are denoted by $\mathcal{L}_I$. For the detail we refer the reader to [1].

Stochastic $\mathfrak{E}$-homogeneous Integral

Definition

1. We call I a *stochastic integral* if $I(E(t)) = E$.
2. The integral is called $\mathfrak{E}$- *homogeneous*, if

$$I(XY(t)) = XI(Y(t)) \text{ for all } X \in \mathfrak{E},\ Y(t) \in \mathcal{L}_I.$$

If I is $\mathfrak{E}$-homogeneous, then every constant vector X in $\mathfrak{E}$ is summable because

$$I(X) = I(XE(t)) = XI(E(t)) = XE = X \in \mathfrak{E}.$$

We also note that a stochastic $\mathfrak{E}$-homogeneous integral I is a projection:

$$\begin{aligned} I^2(X(t)) = I(I(X(t))) &= I([I(X(t)]E(t))) \\ &= [I(X(t))]I(E(t)) = [I(X(t))]E = I(X(t)). \end{aligned}$$

The double integral

Let I and J denote two extended positive vector valued Daniell integrals each of which is stochastic and $\mathfrak{E}$-homogeneous with values in $\mathfrak{E}$, the latter being a universally complete Riesz space. The spaces $\mathcal{L}_I$ and $\mathcal{L}_J$ of summable functions are subspaces of $\mathfrak{E}^S$ and $\mathfrak{E}^T$ respectively.
If we identify the element $X(s) \otimes Y(t) \in \mathcal{L}_I \otimes \mathcal{L}_J$ with the element $X(s)Y(t) \in \mathfrak{E}^{S\times T}$, we find that $\mathcal{L}_I \otimes \mathcal{L}_J$ is a subspace of the Dedekind complete Riesz space $\mathfrak{E}^{S\times T}$. Hence, the Fremlin tensor product $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$ is, in this case, the Archimedean Riesz subspace generated by $\mathcal{L}_I \otimes \mathcal{L}_J$ in the Riesz space $\mathfrak{E}^{S\times T}$.

The double integral (continued)

We now define a bilinear operator

$$b(X(s), Y(t)) = I(X(s))J(Y(t)) \in \mathfrak{E},\ X(s) \in \mathcal{L}_I, Y(t) \in \mathcal{L}_J, \quad (0.1)$$

This is a positive bilinear operator defined on $\mathcal{L}_I \times \mathcal{L}_J$ with values in the f-algebra $\mathfrak{E}$.
Using the universal properties of the Fremlin vector lattice tensor product, there exists a unique positive linear operator $K : \mathcal{L}_I \overline{\otimes} \mathcal{L}_J \to \mathfrak{E}$ with the property that

$$K(X \otimes Y) = I(X)J(Y),\ \text{for all } X \in \mathcal{L}_I,\ Y \in \mathcal{L}_J.$$

We shall show that K is a Daniell integral with initial domain

$$\mathcal{L}_I \overline{\otimes} \mathcal{L}_J \subset \mathfrak{E}^{S \times T}.$$

The double integral (continued)

In exactly the same manner, by interchanging the variable in the above definition of K, we find a unique extension $\tilde{K}$ of the bilinear form $\tilde{b}(Y,X) := J(Y)I(X)$ satisfying

$$\tilde{K}(Y \otimes X) = J(Y)I(X) = I(X)J(Y) = K(X \otimes Y),$$

since the f-algebra $\mathfrak{E}$ is commutative.

The iterated integral

Following Zaanen [4] (see also Rompf and Kersting [3]), we denote by $\mathcal{L}_I * \mathcal{L}_J$ the collection of all elements $U(s,t) \in \mathfrak{E}^{S\times T}$ with the property that for fixed s we have that $U(s,t) \in \mathcal{L}_J$ and $J(U(s,t)) \in \mathcal{L}_I$ i.e.,

$$\mathcal{L}_I * \mathcal{L}_J := \{U(s,t) \in \mathfrak{E}^{S\times T} : I(J(U(s,t))) \in \mathfrak{E}\}.$$

Observe that $\mathcal{L}_I * \mathcal{L}_J$ is a Riesz subspace of $\mathfrak{E}^{S\times T}$ that contains $\mathcal{L}_I \otimes \mathcal{L}_J$ and therefore also $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$. We define the positive operator $I * J$ on $\mathcal{L}_I * \mathcal{L}_J$ by

$$I * J(U) := I(J(U(s,t))) \in \mathfrak{E}.$$

For every element $U(s,t)$ in the tensor product $\mathcal{L}_I \otimes \mathcal{L}_J$ of the form

$$U(s,t) = \sum_{i=1}^{n} X_i(s) \otimes Y_i(t),\ X_i(s) \in \mathcal{L}_I,\ Y_i(t) \in \mathcal{L}_J.$$

we have that

$$
\begin{aligned}
I * J(U(s,t)) &= \sum_{i=1}^{n} I(J(X_i(s))Y_i(t)) \\
&= \sum_{i=1}^{n} I(X_i(s)J(Y_i(t)))(J \text{ is } \mathfrak{E}\text{-homogeneous}) \\
&= \sum_{i=1}^{n} J(Y_i(t))I(X_i(s))(I \text{ is } \mathfrak{E}\text{-homogeneous}) \\
&= \sum_{i=1}^{n} I(X_i(s))J(Y_i(t))(f\text{-algebra is commutative}). \\
&= K(U(s,t)).
\end{aligned}
$$

Hence K and $I * J$ are positive operators on $\mathcal{L}_I \overline{\otimes} \mathcal{L}_J$ that coincide on $\mathcal{L}_I \otimes \mathcal{L}_J$, and so they are equal.

Fubini's theorem

We show that K is a Daniell integral on $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$. Let $Z_n(s,t) \downarrow 0$ for every $(s,t) \in S \times T$, with $Z_n \in \mathcal{L}_I\overline{\otimes}\mathcal{L}_J$. Then for every fixed s $Z_n^s(t) : Z_n(s,t) \downarrow 0$ and so $J(Z_n^s(t)) \downarrow 0$ for every fixed s. But then, $I * J(Z_n(s,t) = I(J(Z_n(s,t))) \downarrow 0$. But, on $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$, $I * J$ is equal to K and so $K(Z_n(s,t)) \downarrow 0$. This shows that $K = I * J$ is a primitive Daniell integral on the initial domain $\mathcal{L}_I\overline{\otimes}\mathcal{L}_J$ and can be extended to the Riesz space of Daniell summable functions using Daniell's extension procedure. This extension process preserves the property that $I * J = K$ and we get Fubini's theorem:

Theorem

Every K-summable function belongs to $\mathcal{L}_I * \mathcal{L}_J$ and moreover, $K(U) = IJ(U)$ for every $I \otimes J$-summable function.

Remarks

1. If $\mathfrak{E}$ is super Dedekind complete, every stochastic $\mathfrak{E}$-homogeneous Daniell integral is a conditional expectation. This is because the integral is order continuous mapping $\mathcal{L}_I$ onto $\mathfrak{E}$, mapping the order unit onto the order unit, and is a projection. To see that it is a mapping onto $\mathfrak{E}$, let $X \in \mathfrak{E}$ Consider the function $E(t)X \in \mathfrak{E}^T$. Then

$$I(E(t)X) = XI(E(t)) = XE = X.$$

2. Our final remark is that if one considers the Bochner integral in a Banach space B which is also a Banach algebra, then the integral is B-homogeneous, for every step function of the form

$$a(t) = \sum_{i=1}^{n} a_i \chi_{E_i}(t),$$

with integral $I(a(t)) = \sum_{i=1}^{n} \mu(E_i) a_i$, satisfies $I(ca(t)) = cI(a(t))$.

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