Irreducible local representations and pure local completely positive and local completely contractive maps of locally C*-algebras

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Outline

- Introduction
- Local operator system
- Stinespring's Theorem
- Irreducible local representations
- Pure maps

Origins

Arveson (1969/1972) uses completely positive maps as the basis of his work on non-commutative dilation theory and non-self-adjoint operator algebras.

- W.B. Arveson, *Subalgebras of C*-algebras*, Acta Math. 123 (1969), 141-224.
- W.B. Arveson, *Subalgebras of C*-algebras II*, Acta Math. 128 (1972), 271-308.

Wittstock (1979) extended Arveson's original result and introduced the notion of operator convexity or matrix convexity, although the methods were difficult and did not extend easily.

- G. Wittstock, *Ein operatwertiger Hahn-Banach satz*, J. Funct. Anal. 40 (1981), 127-150.

Origins

Z.-J. Ruan (1988) provided an axiomatization for operator spaces, known as Ruan's representation theorem: Each (abstract) operator space is completely isometrically isomorphic to a concrete operator space.

- Zhong-Jin Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal., 76 (1988), 217-230.

S. Winkler (1996) proved a version of the bipolar theorem and give a simplified proof of Arveson-Wittstock-Hahn-Banach theorem in even greater generality.

- S. Winkler, *Matrix convexity*, Ph.D. thesis, University of California, Los Angeles, 1996.

Origins

C. J. Webster (1997) developed a theory of "non-commutative locally convex spaces" analogous to the theory of operator spaces, under the title local operator spaces.

- C. Webster, *Local operator spaces and applications*, Ph.D. thesis, University of California, Los Angeles, 1997.

A. Dosiev (2008) introduced a representation theorem for local operator spaces which extends Ruan's representation theorem for operator spaces.

- A. Dosiev, *Local operator spaces, unbounded operators and multinormed C^*-algebras*, J. Funct. Anal., 255(7) (2008), 1724-1760.

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Locally C^*-algebras

Let $\mathcal{A}$ be an unital $*$-algebra with unit $1_{\mathcal{A}}$ and let $(\Lambda, \leq)$ be a directed poset. A family of seminorms $\mathcal{P} := \{p_\lambda \ : \ \lambda \in \Lambda\}$ on $\mathcal{A}$ is called an *upward filtered family* if $\lambda_1 \leq \lambda_2$ in Λ implies that $p_{\lambda_1}(a) \leq p_{\lambda_2}(a)$ for every $a \in \mathcal{A}$.

Definition

A **locally** C^***-algebra** $\mathcal{A}$ is a $*$-algebra together with an upward filtered (saturated) family of C^*-seminorms $\mathcal{P}$ on $\mathcal{A}$ such that $\mathcal{A}$ is complete with respect to the locally convex topology generated by the family $\mathcal{P}$.
We say that $\mathcal{A}$ is a **Frechet locally** C^***-algebra** if the family $\mathcal{P}$ is countable.

Some notations

- $I_\lambda := \{a \in \mathcal{A} \ : \ p_\lambda(a) = 0\}$ an $*-$ ideal;
- The quotient $*$-algebra $\mathcal{A}/I_\lambda$ is a C^*-algebra, denoted by $\mathcal{A}_\lambda$, with the C^*-norm induced by p_λ.
- π_λ denote the canonical quotient $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A}_\lambda$.
- For $n \in \mathbb{N}$, let $M_n(\mathcal{A})$ denotes the set of all $n \times n$ matrices over $\mathcal{A}$. Naturally, $M_n(\mathcal{A})$ is a locally C^*-algebra with the family of seminorms $\{p^n_\lambda : \ \lambda \in \Lambda\}$, defined by $p^n_\lambda([a_{ij}]) = \|\pi^{(n)}_\lambda([a_{ij}])\|_\lambda$ for $[a_{ij}] \in M_n(\mathcal{A})$, where $\pi^{(n)}_\lambda$ stands for the *n-amplification* of the map π_λ.

Locally C^*-algebras

Remark (Arens-Michael)

For $\lambda_1 \leq \lambda_2$ in Λ, there is a canonical $*$-homomorphism $\pi_{\lambda_1\lambda_2} : \mathcal{A}_{\lambda_2} \to \mathcal{A}_{\lambda_1}$, $\pi_{\lambda_1\lambda_2}(a + I_{\lambda_2}) = a + I_{\lambda_1}$ such that $\pi_{\lambda_1\lambda_2} \circ \pi_{\lambda_2} = \pi_{\lambda_1}$. Then one can identify $\mathcal{A}$ as the inverse limit of the projective system $\{\mathcal{A}_{\lambda_1}, \pi_{\lambda_1\lambda_2} : \lambda_1, \lambda_2 \in \Lambda\}$ of C^*-algebras.

Positive elements

Definition

Let $\mathcal{A}$ be a topological $*$-algebra . An element $a \in \mathcal{A}$ is called

- **hermitian** (or **self-adjoint**) if $a^* = a$.
- **positive** and we write $a \geq 0$ if it is hermitian and $sp_{\mathcal{A}}(a) \subseteq [0, \infty) \Leftrightarrow (\exists) b \in \mathcal{A}$ such that $a = b^* b$.

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Local positivity

Definition

An element $a \in \mathcal{A}$ is called **local hermitian** if $a = a^* + x$ for some $x \in \mathcal{A}$ such that $p_\lambda(x) = 0$ for some $\alpha \in \Lambda$. An element $a \in \mathcal{A}$ is called **local positive** if $a = b^*b + x$ for some $b, x \in \mathcal{A}$ such that $p_\lambda(x) = 0$ for some $\lambda \in \Lambda$.
In this case, we say that a is λ-**hermitian** and λ-**positive**, respectively. We denote by $a \geq_\lambda 0$ the fact that a is λ-positive.

Remark

- $a \geq_\lambda 0$ in $\mathcal{A}$ if and only if $\pi_\lambda(a) \geq 0$ in the C^*-algebra $\mathcal{A}_\lambda$.
- a is hermitian (respectively, positive) if and only if it is λ-**hermitian** (respectively, λ-**positive**) for every $\lambda \in \Lambda$.

Local Operator System

Definition

A **local operator system** in $\mathcal{A}$ is an unital self-adjoint linear subspace of $\mathcal{A}$.

Definition

An element a in a local operator system S is **local positive** if a is local positive in $\mathcal{A}$.

Local maps

Consider another locally C^*-algebra $\mathcal{B}$ with the associated family of seminorms $\{q_l : \ l \in \Omega\}$, and let S_1 and S_2 be local operator systems in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition

A linear map $\phi : S_1 \to S_2$ is called

- **local positive** if for each $l \in \Omega$, there exists $\lambda \in \Lambda$ such that $\phi(a) \geq_l 0$ whenever $a \geq_\lambda 0$ in S_1, and $\phi(a) =_l 0$ if $a =_\lambda 0, \ a \in S_1$.
- **local bounded** if for each $l \in \Omega$ there exists an $\lambda \in \Lambda$ and $C_{l,\lambda} > 0$ such that $q_l(\phi(a)) \leq C_{l,\lambda} p_\lambda(a)$ for all $a \in S_1$.
- **local contractive** if $C_{l,\lambda} > 0$ can be chosen above to be 1.
- **local completely bounded** (**local CB-map**) if for each $l \in \Omega$, there exist $\lambda \in \Lambda$ and $C_{l,\lambda} > 0$ such that $q_l^n([\phi(a_{ij})]) \leq C_{l,\lambda} p_l^n([a_{ij}])$, for every $n \in \mathbb{N}$.
- **local completely contractive (local CC-map)** if $C_{l,\lambda} = 1$ above.
- **local completely positive (local CP-map)** if for each $l \in \Omega$, there exists $\lambda \in \Lambda$ such that $\phi^{(n)}([a_{ij}]) \geq_l 0$ in $M_n(S_2)$ whenever $[a_{ij}] \geq_\lambda 0$ in $M_n(S_1)$.

Quantized domain

Definition

A **quantized domain** in $\mathcal{H}$ is a triple $\{\mathcal{H}, \mathcal{E}, \mathcal{D}\}$, where $\mathcal{E} := \{H_l : \ l \in \Omega\}$ is an upward filtered family of closed subspaces such that the union space $\mathcal{D} := \bigcup_{l \in \Omega} H_l$ is dense in $\mathcal{H}$. *In short*, we say that $\mathcal{E}$ *is a quantized domain in* $\mathcal{H}$ *with its union space* $\mathcal{D}$. A quantized domain $\mathcal{E}$ is called a **quantized Frechet domain** if $\mathcal{E}$ is a countable family.

Remark

The quantized family $\mathcal{E} := \{H_l : \ l \in \Omega\}$ determines an upward filtered family $\{P_l : \ l \in \Omega\}$ of projections in $B(\mathcal{H})$, where P_l is the orthogonal projection of $\mathcal{H}$ onto the closed subspace H_l.

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Non-commutative continuous functions

Definition

The set of all **non-commutative continuous functions** on a quantized domain $\mathcal{E}$ is defined as

$$C_{\mathcal{E}}(\mathcal{D}) := \{ T \in L(\mathcal{D}) : \ TP_l = P_l T P_l \in B(\mathcal{H}), \ \forall \ l \in \Omega \},$$

where $L(\mathcal{D})$ denotes the set of all linear operators on the linear subspace $\mathcal{D}$.

Note that $C_{\mathcal{E}}(\mathcal{D})$ is an algebra and if $T \in L(\mathcal{D})$, then

$$T \in C_{\mathcal{E}}(\mathcal{D}) \Leftrightarrow T(H_l) \subseteq H_l \textit{ and } T \restriction_{H_l} \in B(H_l) \textit{ for all } l \in \Omega.$$

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Beyond $B(\mathcal{H})$

Definition

The $*$-**algebra of all non-commutative continuous functions** on a quantized domain $\mathcal{E}$ is defined by

$$C^*_{\mathcal{E}}(\mathcal{D}) := \{T \in C_{\mathcal{E}}(\mathcal{D}) : \; P_l T \subseteq T P_l, \; \forall \, l \in \Omega\}.$$

The adjoint

If $T \in L(\mathcal{D})$, then T is a densely defined linear operator on $\mathcal{H}$. The **adjoint** of T is a linear map $T^\bigstar : dom(T^\bigstar) \subseteq \mathcal{H} \to \mathcal{H}$, where

$$dom(T^\bigstar) := \{\xi \in \mathcal{H} : \; \eta \to \langle T\eta, \xi \rangle \textit{ is continous for every } \eta \in dom(T)\}$$

such that $\langle T\eta, \xi \rangle = \langle \eta, T^\bigstar \eta \rangle$ for all $\xi \in dom(T^\bigstar)$ and $\eta \in dom(T)$.

Beyond $B(\mathcal{H})$

Remark

- $C^*_{\mathcal{E}}(\mathcal{D})$ is an unital subalgebra of $C_{\mathcal{E}}(\mathcal{D})$.
- If $T \in L(\mathcal{D})$, then $T \in C^*_{\mathcal{E}}(\mathcal{D})$ if and only if $T(H_l) \subseteq H_l$, $T\left(H_l^{\perp} \cap \mathcal{D}\right) \subseteq H_l^{\perp} \cap \mathcal{D}$ and $T\restriction_{H_l} \in B(H_l)$ for all $l \in \Omega$.
- If $T \in C^*_{\mathcal{E}}(\mathcal{D})$, then

$$\mathcal{D} \subseteq dom(T^{\bigstar}),\ \ T^{\bigstar}(\mathcal{D}) \subseteq \mathcal{D} \ and \ T^* = T^{\bigstar}\restriction_{\mathcal{D}} \in C^*_{\mathcal{E}}(\mathcal{D})$$

  .
- The correspondence $T \mapsto T^* = T^{\bigstar}\restriction_{\mathcal{D}} \in C^*_{\mathcal{E}}(\mathcal{D})$ is an involution on $C^*_{\mathcal{E}}(\mathcal{D})$.

Beyond $B(\mathcal{H})$

For each $l \in \Omega$, the map $q_l : C^*_{\mathcal{E}}(\mathcal{D}) \to [0,\infty)$ defined by $q_l(T) := \|T\|_l := \|T\restriction_{H_l}\|$ is a C^*-seminorm on $C^*_{\mathcal{E}}(\mathcal{D})$.

Remark

- $C^*_{\mathcal{E}}(\mathcal{D})$ is a locally C^*-algebra with respect to the family of C^*-seminorms $\{q_l : l \in \Omega\}$.
- If $\mathcal{E} = \{\mathcal{H}\}$, then $C^*_{\mathcal{E}}(\mathcal{D}) = B(\mathcal{H})$.

$\mathcal{CPCC}_{loc}(S, C^*_{\mathcal{E}}(\mathcal{D}))$ stands for the class of all local completely positive and local completely contractive maps from a local operator system S to $C^*_{\mathcal{E}}(\mathcal{D})$.

Local representations

Definition

Let $\mathcal{A}$ be an unital locally C^*-algebra with the topology defined by the family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$. A **local representation** of $\mathcal{A}$ on a quantized domain $\{\mathcal{H};\mathcal{E};\mathcal{D}\}$ with $\mathcal{E}=\{H_l\}_{l\in\Omega}$, is a $*$-homomorphism $\pi:\mathcal{A}\to C^*_{\mathcal{E}}(\mathcal{D})$ with the property that for each $l\in\Omega$, there exist $\lambda\in\Lambda$ and $M_\lambda>0$ such that $\|\pi(a)\|_l\le M_\lambda p_\lambda(a)$ for all $a\in\mathcal{A}$. If $M_\lambda=1$, we say that π is a **local contractive representation.**

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Local representations

Definition

We say that two local representations $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ and $\tilde{\pi} : \mathcal{A} \to C^*_{\tilde{\mathcal{E}}}(\tilde{\mathcal{D}})$ are **unitarily equivalent** if there exists a unitary operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $U(H_l) \subseteq \tilde{H}_l$ for all $l \in \Omega$ and $U\pi(a) \subseteq \tilde{\pi}(a)U$.

Definition

A local representation $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ is called **non-degenerate** if $[\pi(\mathcal{A})\mathcal{D}] = \mathcal{H}$.

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Non-degenerate local representations

Proposition

Let $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local representation. Then π is non-degenerate if and only if $[\pi(\mathcal{A})H_l] = H_l, \ \forall \ l \in \Omega$.

Representation theorem for locally C^*-algebras

Theorem (Dosiev-2008)

Let $\mathcal{A}$ be an unital locally C^*-algebra. Then there is a local isometrical $*$-homomorphism $\mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ for some quantized domain $\mathcal{E}$ with its union space $\mathcal{D}$.

Stinespring's theorem for local CP-maps

Theorem (Dosiev-2008)

Let $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$. Then there exists a Hilbert space H^ϕ and a quantized domain $\mathcal{E}^\phi := \left\{H^\phi_l : \ l \in \Omega\right\}$ in H^ϕ with its union space $\mathcal{D}^\phi$, a contraction $V_\phi : H \to H^\phi$, and an unital local contractive $*$-homomorphism $\pi_\phi : \mathcal{A} \to C^*_{\mathcal{E}^\phi}(\mathcal{D}^\phi)$ such that

$$\phi(a) \subseteq V^*_\phi \pi_\phi(a) V_\phi \textit{ and } V_\phi(H_l) \subseteq H^\phi_l$$

for every $a \in \mathcal{A}$ and $l \in \Omega$. Moreover, if $\phi(1_{\mathcal{A}}) = 1_{\mathcal{D}}$, then V_ϕ is an isometry.

Minimal Stinespring representation

B. R. Bhat, A. Ghatak, S. K. Pamula (2021) introduced a suitable notion of minimality for Stinespring's theorem for local CP-maps on locally C^*-algebras to ensure uniqueness up to unitary equivalence for the associated representation.

- B. R. Bhat, A. Ghatak, S. K. Pamula, *Stinespring's theorem for unbounded operator valued local completely positive maps and its applications*, Indagationes Mathematicae, 32(2) (2021), 547-578.

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Minimal Stinespring representation

Definition (Bhat-Ghatak-Pamula-2021)

Any triple $\left(\pi_\phi, V_\phi, \left\{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\right\}\right)$ that satisfies the conditions of the previous theorem is called a **Stinespring representation** for ϕ. A Stinespring representation $\left(\pi_\phi, V_\phi, \left\{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\right\}\right)$ of ϕ is called **minimal** if $H_l^\phi = [\pi_\phi V_\phi H_l]$ for every $l \in \Omega$.

Proposition (Bhat-Ghatak-Pamula-2021)

Let $\left(\pi_\phi, V_\phi, \left\{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\right\}\right)$ be a Stinespring representation of $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_\mathcal{E}(\mathcal{D}))$. Then there is a minimal Stinespring representation $\left(\tilde{\pi}_\phi, \tilde{V}_\phi, \left\{\tilde{H}^\phi; \tilde{\mathcal{E}}^\phi; \tilde{\mathcal{D}}^\phi\right\}\right)$ for ϕ such that $\tilde{\mathcal{D}}^\phi \subseteq \mathcal{E}^\phi$ and $\tilde{H}^\phi = \left[\tilde{\pi}_\phi(\mathcal{A})\tilde{V}_\phi\mathcal{D}\right]$.

The starting point of our questions

- **C. S. Arunkumar**, *Local boundary representations of locally C^*-algebras*, Journal of Mathematical Analysis and Applications, 515(2) (2022), 126416.

Definition

Let $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ be a representation. The **commutant** of $\pi(\mathcal{A})$ is defined as $\pi(\mathcal{A})' := \{T \in B(\mathcal{H}) : \; T\pi(a) \subseteq \pi(a)T\}$.

Definition

A representation $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ is said to be **irreducible** if $\pi(\mathcal{A})' \cap C^*_{\mathcal{E}}(\mathcal{D}) = \mathbb{C}I_{\mathcal{D}}$.

The starting point of our questions

Remark

In fact, the author remained in the "classic" case on $B(\mathcal{H})$ as it has been pointed out by M. Joița.

- M. Joița, *The Choquet boundary for a local operator system*, preprint.

Question

What does a suitable notion of irreducible representation look like?

Irreducible local representations

Definition

The **center** of $C^*_{\mathcal{E}}(\mathcal{D})$ is the locally von Neumann algebra $\mathcal{Z}\left(C^*_{\mathcal{E}}(\mathcal{D})\right)$ generated by the family $\{P_l : \ l \in \Omega\}$.

Definition

Let $\{\mathcal{H};\mathcal{E};\mathcal{D}\}$ be a quantized domain, $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local representation of $\mathcal{A}$ and let $\pi(\mathcal{A})' := \{T \in C^*_{\mathcal{E}}(\mathcal{D}) : \quad T\pi(a) \subseteq \pi(a)T\}$. We say that $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ is **irreducible** if $\pi(\mathcal{A})' = \mathcal{Z}\left(C^*_{\mathcal{E}}(\mathcal{D})\right)$.

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Unitary equivalence

Proposition

Let $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ and $\tilde{\pi} : \mathcal{A} \to C^*_{\tilde{\mathcal{E}}}(\mathcal{D})$ be two local representations of $\mathcal{A}$. If π and $\tilde{\pi}$ are unitarily equivalent and π is irreducible, then $\tilde{\pi}$ is irreducible.

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The Frechet algebra $C_{\mathcal{E}}(\mathcal{D})$

Let $C_{\mathcal{E}}(\mathcal{D})$ be a Frechet algebra, where $\mathcal{E} = \{H_n\}_{n\geq 1}$. Let $\{P_n\}_{n\geq 1}$ be the projection net associated to $\mathcal{E}$, and let $S_n := (I - P_{n-1}) P_n$ be the projection onto the subspace $H_{n-1}^{\perp} \cap H_n, n \geq 2$, where for $n = 1$ we set $S_1 = P_1$.

Proposition (Dosiev-2008)

If $T \in C_{\mathcal{E}}(\mathcal{D})$, then it has a triangular matrix representation
$T = \sum_{m=1}^{\infty} \sum_{k=1}^{m} S_k T S_m = \begin{bmatrix} T_{11} & T_{12} & \cdots \\ 0 & T_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$. Moreover, if $T \in C_{\mathcal{E}}^{*}(\mathcal{D})$, then it has a diagonal representation $T = \sum_{m=1}^{\infty} S_m T S_m$.

Irreducible local representations

Definition

Let $\{\mathcal{H};\mathcal{E};\mathcal{D}\}$ be a Frechet quantized domain (i.e., $\mathcal{E}=\{H_n\}_{n\geq 1}$) and let $\pi:\mathcal{A}\to C^*_{\mathcal{E}}(\mathcal{D})$ be a local representation. We define the maps $\pi_n:\mathcal{A}\to B\left(H_{n-1}^{\perp}\cap H_n\right),\ \pi_n(a):=\pi(a)\restriction_{H_{n-1}^{\perp}\cap H_n}$ for $n>1$ and $\pi_1:\mathcal{A}\to B\left(H_1\right),\pi_1(a):=\pi(a)\restriction_{H_1}$.

Theorem

Let $\{\mathcal{H};\mathcal{E};\mathcal{D}\}$ be a Frechet quantized domain (i.e., $\mathcal{E}=\{H_n\}_{n\geq 1}$) and let $\pi:\mathcal{A}\to C^*_{\mathcal{E}}(\mathcal{D})$ be a local representation. Then π is irreducible if and only if for each $n\geq 1$, π_n is an irreducible representation.

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Irreducible local representations

Corollary

Let $\{\mathcal{H};\mathcal{E};\mathcal{D}\}$ be a Frechet quantized domain and let $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local representation. If π is irreducible, then π is non-degenerate.

Question

What about the general situation?

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The starting point of our questions

- **C. S. Arunkumar**, *Local boundary representations of locally C^*-algebras*, Journal of Mathematical Analysis and Applications, 515(2) (2022), 126416.

Definition

A map $\varphi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$ is called **pure** if for any map $\psi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$ such that $\varphi - \psi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$, then there is a scalar $t \in [0, 1]$ such that $\psi = t\varphi$.

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The starting point of our questions

Remark

Again the author stayed in the "classic" case on $B(\mathcal{H})$ as it has been pointed out by M. Joița.

- M. Joița, *The Choquet boundary for a local operator system*, preprint.

Question

How can we correctly define the notion of a "pure map"?

Pure maps

Definition

Let $\varphi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$. We say that φ is **pure** if for each $\psi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$ with $\varphi - \psi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$ we have that for

$$\forall\ I \in \Omega:\ \psi(a) = \sum_{I} \lambda_I Q_I \varphi(a)$$

for some $\lambda_I \geq 0$ and for $\forall\ a \in \mathcal{A}$, where $Q_I := P_{I_1} P_{I_2} \cdot \ldots \cdot P_{I_n} \restriction_{\mathcal{D}}$.

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Pure maps

Proposition

Let $\varphi \in \mathcal{CPCC}_{loc}(\mathcal{A}, C^*_{\mathcal{E}}(\mathcal{D}))$. Then φ is pure if and only if its minimal Stinespring representation is irreducible.

Thank you for your attention !