# A representation of variable exponent spaces $L^{p(\cdot)}(\Omega)$ for infinite measures

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POSITIVITY XI

Joint work with Cesar Ruiz and Mauro Sanchiz

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measurable $p(\cdot) : (\Omega, \Sigma, \mu) \mapsto [1, \infty)$,

$L^{p(\cdot)}(\Omega)$ Variable exponent (or Nakano ) space is the space of all scalar measurable f. s.t. the modular

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$R_{p(\cdot)}$ is a lattice-isometric invariant (P.Poitevin,Y.Raynaud 2.008)

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**Proposition ( F.H., C. Ruiz 2.013)**

*$L^{p(\cdot)}(\Omega)$ for $\mu(\Omega) < \infty$ and $p^+ < \infty$.*

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$L^{p(\cdot)}(\Omega)$ has a $\ell_q$-complemented sublattice for $q \in R_{p(\cdot)}$

orthogonal pojection $T_A$ for suitable $(A_k)$

$$T_A(f)(t) = \sum_{k=1}^{\infty} \left( \int_{A_k} \frac{f(s)}{\mu(A_k)^{\frac{1}{p^*(s)}}} d\mu(s) \right) \frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}},$$

$\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1$

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$L^p(\Omega, \Sigma, \mu)$ is lattice-isomorphic to $L^p((0,1), \mathcal{B}, \lambda)$ , $1 \leq p < \infty$

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• there is Orlicz s. $L^\varphi(\mu)$ , $\mu$ infinite, non-isomorphic to any Orlicz $L^\psi(0,1)$

$$L^{x^p \wedge x^q}(0,\infty) = L_p + L_q(0,\infty) \text{ , } 1 < p < q < 2$$

there is no $L^\psi(0,1)$ s.t. $L^\psi(0,1) \simeq L_p + L_q(0,\infty)$

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why ?

## Theorem

*$L^{p(\cdot)}(\Omega, \mu)$ on non-atomic separable $\sigma$-finite measure s. $(\Omega, \Sigma, \mu)$. There exists exponent $\widehat{p(\cdot)}$ on $(0, \infty)$ s,t.*

$$L^{p(\cdot)}(\Omega, \mu) \cong L^{\widehat{p}(\cdot)}(0, \infty)$$

*The operator $T_\phi : L^{p(\cdot)}(\Omega) \to L^{\widehat{p}(\cdot)}(0, \infty)$ $T_\phi(f) := \widehat{f^+} - \widehat{f^-} = \widehat{f}$, for $f = f^+ - f^-$, is an isomodular lattice-isomorphism,*

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proof (hints):
extension of Caratheodory Th. for $\sigma$-finite measures

**Theorem (Carathéodory, 1946)**

$(\Omega, \Sigma, \mu)$ *non-atomic separable probability measure space,* $((0,1), \mathcal{B}, \lambda)$ *.*
*There exists an isomorphism* $\phi : \Sigma/\mathcal{N}_\mu \to \mathcal{B}/\mathcal{N}_\lambda$ *preserving measures*

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simple $S = \sum_{n\in\Delta} a_n \chi_{A_n} \in L_0(\Omega)$, $\widehat{S} := \sum_{n\in\Delta} a_n \chi_{\phi(A_n)}$, simple in $L_0(0,\infty)$ .
For $0 \leq f \in L_0(\Omega)$, define

$$\widehat{f} := \lim_{n\to\infty} \widehat{S_n}$$

in $L_0(0,\infty)$, where positive increasing simple $S_n \nearrow f$, since $(\widehat{S_n})$ is also increasing a.e-$\lambda$.
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this map $\widehat{}$ extends to general measurable f. It is good definition of $\widehat{p}(\cdot)$ with the properties

$$\widehat{f+g} = \widehat{f} + \widehat{g}$$

,

$$\widehat{f\,\chi_A} = \widehat{f}\,\chi_{\phi(A)} \qquad \widehat{f^{\,p(\cdot)}} = \widehat{f}^{\,\widehat{p(\cdot)}}$$

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Let $\tau : (0,1) \to (0,\infty)$ diffeomorphism (f.i. $\tau(s) = \tan(\frac{\pi}{2}s)$).

$$q(\cdot) : (0,1) \to [1,\infty) \; q(s) := p(\tau(s))$$

$$\int_0^\infty |f(t)|^{p(t)} dt = \int_0^1 |f(\tau(s))|^{q(s)} |\tau'(s)| ds = \int_0^1 \left( \left| f(\tau(s)) \tau'(s)^{\frac{1}{q(s)}} \right| \right)^{q(s)} ds.$$

The operator $T_\tau : L^{p(\cdot)}(0,\infty) \to L^{q(\cdot)}(0,1) \quad T_\tau f(s) = \left| \tau'(s)^{\frac{1}{q(s)}} \right| f(\tau(s))$, is an (isomodular) lattice-isomorphism.

The inverse operator $T_\tau^{-1}$

$$(T_\tau^{-1} g)(t) = \left| (\tau')^{-1}(t)^{\frac{1}{p(t)}} \right| g(\tau^{-1}(t)) = (T_{\tau^{-1}} g)(t).$$

$$L^{p(\cdot)}(\Omega) \approx L^{\widehat{p}(\cdot)}(0,\infty) \approx L^{q(\cdot)}(0,1)$$

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**Corollary**

*Every $L^{p(\cdot)}(\Omega)$ on non-atomic separable $\sigma$-finite measure space $(\Omega,\mu)$ is (isomodular) lattice-isomorphic to v.e. $L^{q(\cdot)}(0,1)$ with $R_{p(\cdot)} = R_{q(\cdot)}$.*

$$\int_\Omega |f(t)|^{p(t)} d\mu(t) = \int_0^1 |f(s)|^{q(s)} d\lambda(s)$$

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$L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$.

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$L^{p(\cdot)}(\Omega)$ subprojective $\iff 2 \leq p^- \leq p^+ < \infty$

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operator $T : L^{p(\cdot)}(\Omega) \mapsto L^{p(\cdot)}(\Omega)$ , $p^+ < \infty$

is strictly singular (or Kato) $\iff$ is $\ell_q$-singular for $q \in R_{p(\cdot)} \cup \{2\}$

( L. Weis for $L_p$-spaces )

$p^+ < \infty$ , infinite $\mu$

$S \subset L^{p(\cdot)}(\Omega)$ , relatively weakly compact $\iff$

$$\lim_{\lambda \to 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{p(t)} d\mu = 0$$

$$\lim_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n \cap \Omega_1} |f(t)| d\mu = 0.$$

$\Omega_1 := p^{-1}\{1\}$

# Weak compactness and weak convergence

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$(f_n) \subset L^{p(\cdot)}(\Omega)$ weakly converg. $f \in L^{p(\cdot)}(\Omega)$ $\iff$

(i) $\lim_n \int_A f_n d\mu = \int_A f d\mu$ for $\mu(A) < \infty$,

(ii) $\lim_{\lambda \to 0} \sup_n \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda(f_n - f)|^{p(t)} d\mu = 0$,

(iii) $\lim_{A_k \searrow \emptyset} \sup_{n \in \mathbb{N}} \int_{A_k \cap \Omega_1} |f_n(t) - f(t)| d\mu = 0$.

Some questions:

Fixed $q > 2$

- Is there $L^{p(\cdot)}(\Omega)$ with $\mu(p^{-1}\{q\}) = 0$ s.t. $L^{p(\cdot)}(\Omega) \supsetsim L^q$ ?

- criteria for that ?

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( This is true for symmetric function spaces )

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THANK YOU VERY MUCH ¡