A generalization of Riesz* homomorphisms in order unit spaces

Florian Boisen

Dresden University of Technology, Germany

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Joint work with A. Kalauch, J. Stennder, O. van Gaans

Pre-Riesz spaces

Definition

An ordered vector space X is called *pre-Riesz* if there exists a vector lattice Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y, i.e.,

$$\forall y \in Y: \quad y = \inf\{i(x);\, x \in X, i(x) \geq y\}.$$

The pair (Y, i) is called a *vector lattice cover* of X. If Y is the smallest vector lattice[1] that contains $i[X]$, then (Y, i) is called the *Riesz completion* of X.

- The Riesz completion of X is unique up to order isomorphism.

[1] with respect to inclusion

Examples for pre-Riesz spaces

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Every vector lattice is a pre-Riesz space.
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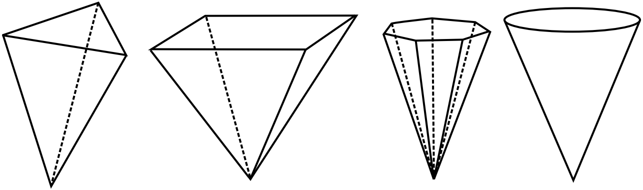
- $\mathrm{C}^n[a, b]$, $\mathrm{P}^n[a, b]$
- $\mathrm{L}^{\mathrm{r}}(X, Y)$ with X directed and Y Archimedean
- Finite-dimensional spaces X with closed positive cone K and $\mathrm{int}(K) \neq \varnothing$.



Order unit spaces

Let X be an ordered vector space. An element $u \in X$, $u > 0$ is called an *order unit* if

$$\forall x \in X \, \exists \lambda \in (0, \infty) : \quad x \in [-\lambda u, \lambda u].$$

Then X is directed.

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- For example, if (X, K) is an ordered normed space with $\mathrm{int}(K) \neq \emptyset$, then u is an order unit $\iff$ $u \in \mathrm{int}(K)$.

Riesz* homomorphisms

Definition

Let X and Y be pre-Riesz spaces with respective Riesz completions (X^ρ, i_X) and (Y^ρ, i_Y). A linear map $T\colon X \to Y$ is called a *Riesz* homomorphism* if there exists a lattice homomorphism $T^\rho\colon X^\rho \to Y^\rho$ such that $T^\rho \circ i_X = i_Y \circ T$.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X^\rho & \xrightarrow{T^\rho} & Y^\rho \end{array}$$

- The extension T^ρ is unique.

Let (X, K) be an order unit space with order unit u. The set

$$\Sigma := \{\varphi \in K'; \, \varphi(u) = 1\}$$

is a weakly-* compact base of K'. Define $\Lambda := \operatorname{ext}(\Sigma)$ and let $\overline{\Lambda}$ be the weak-*-closure of Λ in Σ.

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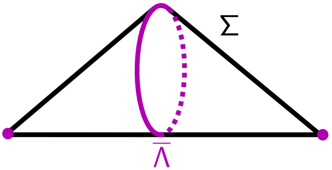
- For example, consider the ordered vector space $(\mathbb{R}^4, K)$ where the dual base Σ (as a subset of $\mathbb{R}^3$) is:



Theorem (Van Haandel, 1993)

Let X and Y be pre-Riesz spaces. A linear map $T\colon X \to Y$ is a Riesz homomorphism if and only if*

$$\forall \varnothing \neq F \subseteq X \text{ finite}: \quad T\left[F^{\mathrm{u}\ell}\right] \subseteq T[F]^{\mathrm{u}\ell}.$$

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- 2018: A gap in van Haandel's proof for this claim has been found.
- 2023: A counterexample for van Haandel's claim has been found.

Mild Riesz* homomorphisms

Definition

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Immediate properties:

- Riesz* $\implies$ mild Riesz* $\implies$ positive
- If X and Y are vector lattices, then

$$\{\text{mild Riesz* hom.}\} = \{\text{lattice hom.}\} = \{\text{Riesz* hom}\}.$$

A mild Riesz* hom. that is not a Riesz* hom.

Let B be the closed unit ball in $\mathbb{R}^2$ and $\mathrm{Aff}(B)$ the space of affine functions $B \to \mathbb{R}$ endowed with the pointwise order.

[2] $\mathrm{ev}_p\colon \mathrm{Aff}(B) \to \mathbb{R}, f \mapsto f(p)$

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- $\overline{\Lambda} = \{\mathrm{ev}_p;\ p \in \overline{\mathrm{ext}(B)} = \partial B\}$.
- ev_p is mild Riesz* $\iff \forall f, g \in \mathrm{Aff}(B)$:

$$f(p) > g(p)^+ \implies \exists q \in \overline{\mathrm{ext}(B)} = \partial B : f(q) > g(q)^+ \quad (1)$$

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- $\implies$ For a mild Riesz* hom. that is not a Riesz* hom., we need to find points $p \in \mathrm{int}(B)$ that satisfy (1).

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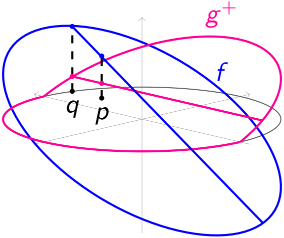

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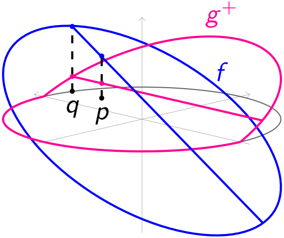

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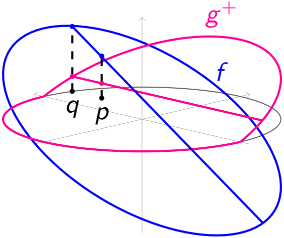

- For every $p \in B$, ev_p is a mild Riesz* homomorphism!
- Every positive linear functional on $\mathrm{Aff}(B)$ is a mild Riesz* homomorphism.

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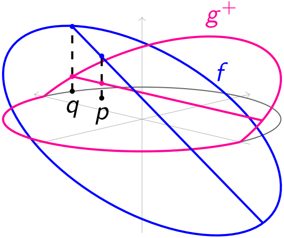

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What we know so far

	Riesz*	$\subseteq$	mild Riesz*	$\subseteq$	K'
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Mild Riesz* homomorphisms on three-dimensional spaces

Theorem (B., Kalauch, Stennder, van Gaans, 2023)

Let X be a three-dimensional order unit space.

- Σ *is strictly convex* $\implies$ *Every positive linear functional on* X *is a mild Riesz* homomorphism.*
- Σ *is not strictly convex* $\implies$ *Every mild Riesz* homomorphism* $X \to \mathbb{R}$ *is a Riesz* homomorphism.*

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Question

Can this be generalized to finite-dimensional order unit spaces?

Mild Riesz* homomorphisms on finite-dimensional spaces

Theorem (B., Kalauch, Stennder, van Gaans, 2023)

Let X be a finite-dimensional order unit space. If every one-dimensional face of Σ is contained in $\overline{\Lambda}$, then every positive linear functional on X is a mild Riesz homomorphism.*

Mild Riesz* homomorphisms on finite-dimensional spaces

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Let X be a finite-dimensional order unit space. If every one-dimensional face of Σ is contained in $\overline{\Lambda}$, then every positive linear functional on X is a mild Riesz homomorphism.*

Questions

- What can happen if Σ has a one-dimensional face that is not entirely contained in $\overline{\Lambda}$?
- Is this also true in infinite dimensions?

And operators?

Theorem (B. Kalauch, Stennder, van Gaans, 2023)

Let X and Y be order unit spaces, where every mild Riesz homomorphism $X \to \mathbb{R}$ is a Riesz* homomorphism. Then every mild Riesz* homomorphism $X \to Y$ is a Riesz* homomorphism.*

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Thank you :)