Free Dual Banach Lattices

Enrique García-Sánchez

Instituto de Ciencias Matemáticas, Madrid

Positivity XI, Ljubljana, July 13 2023

Joint work with Pedro Tradacete.

Grant CEX2019-000904-S-21-3 funded by

MINISTERIO DE CIENCIA E INNOVACIÓN

UNIÓN EUROPEA
Fondo Social Europeo
El FSE invierte en tu futuro

AGENCIA ESTATAL DE INVESTIGACIÓN

Overview

1. Preliminaries
2. $\mathrm{FBL}^{(p)}[E \quad]$ vs $\mathrm{FBL}^{(p)}[E]$
3. Free dual Banach lattices

Overview

1. Preliminaries

2. $\mathrm{FBL}^{(p)}[E\quad]$ vs $\mathrm{FBL}^{(p)}[E]$

3. Free dual Banach lattices

Free objects

Definition

Let C be a subcategory of D, and A an object in D.

Free objects

Definition

Let C be a subcategory of D, and A an object in D. A **free object over A in the category C** is a pair (F_A, j) composed by an object F_A in the category C and a morphism $j : A \to F_A$ in D

Free objects

Definition

Let C be a subcategory of D, and A an object in D. A **free object over A in the category C** is a pair (F_A, j) composed by an object F_A in the category C and a morphism $j : A \ ! \ F_A$ in D such that for every object B in C

Free objects

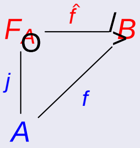
Definition

Let C be a subcategory of D, and A an object in D. A **free object over A in the category** C is a pair (F_A, j) composed by an object F_A in the category C and a morphism $j : A \to F_A$ in D such that for every object B in C and every morphism $f : A \to B$ in D

Free objects

Definition

Let C be a subcategory of D, and A an object in D. A **free object over A in the category C** is a pair (F_A, j) composed by an object F_A in the category C and a morphism $j : A \to F_A$ in D such that for every object B in C and every morphism $f : A \to B$ in D there exists a unique morphism $\hat{f} : F_A \to B$ in C making the following diagram commutative.



Examples

Examples

- the free group over a set,

Examples

- the free group over a set,
- the free Banach space over a set

Examples

- the free group over a set,
- the free Banach space over a set ($\ell_1(A)$),

Examples

- the free group over a set,
- the free Banach space over a set ($\ell_1(A)$),
- the Stone–Čech compactification of a topological space,

Examples

- the free group over a set,
- the free Banach space over a set ($\ell_1(A)$),
- the Stone–Čech compactification of a topological space,
- the Lipschitz-free Banach space over a metric space,

Examples

- the free group over a set,
- the free Banach space over a set ($\ell_1(A)$),
- the Stone–Čech compactification of a topological space,
- the Lipschitz-free Banach space over a metric space,

  ...

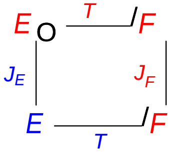
Free dual over a Banach space

The bidual space E^{**} of a Banach space E, together with the canonical embedding $J_E : E \to E^{**}$, can be understood as the **free dual Banach space over a Banach space** E:

Free dual over a Banach space

The bidual space E^{**} of a Banach space E, together with the canonical embedding $J_E : E \to E^{**}$, can be understood as the **free dual Banach space over a Banach space E**:

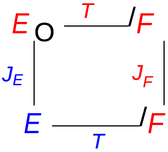
Every linear and bounded operator $T : E \to F^*$, with F^* a dual Banach space, can be extended to an adjoint, linear and bounded operator $J_F^* \circ T^{**} : E^{**} \to F^*$ such that $T = J_F^* \circ T^{**} \circ J_E$ and $\|T\| = \|J_F^* \circ T^{**}\|$.



Free dual over a Banach space

The bidual space E of a Banach space E, together with the canonical embedding $J_E : E \to E$, can be understood as the **free dual Banach space over a Banach space** E:

Every linear and bounded operator $T : E \to F$, with F a dual Banach space, can be extended to an adjoint, linear and bounded operator $J_F \; T : E \to F$ such that $T = J_F \; T \; J_E$ and $\|T\| = \|J_F \; T\|$.



Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \| \|)$ equipped with a partial order satisfying:

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order satisfying:

- if $x \le y$, then $x + z \le y + z$ for every $z \in X$;
- if $x \le y$, then $\lambda x \le \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);
- if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);
- if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

Definition

An operator $T : X \to Y$ between two Banach lattices is called a **lattice homomorphism** if it is linear and preserves the lattice operations.

Definition

A Banach lattice X is said ***p*-convex** for $1 \le p \le \infty$ if there exists a constant $M \ge 1$ such that for any $x_1, \ldots, x_n \in X$ the inequality

$$\Big\| \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}} \Big\| \le M \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{\frac{1}{p}}$$

holds,

p-convex Banach lattices

Definition

A Banach lattice X is said **p-convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that for any $x_1, \ldots, x_n \in X$ the inequality

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}$$

holds, where

$$\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^n a_i x_i : \left(\sum_{i=1}^n |a_i|^{p'} \right)^{\frac{1}{p'}} \leq 1 \right\} \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

p-convex Banach lattices

Definition

A Banach lattice X is said **p-convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that for any $x_1, \ldots, x_n \in X$ the inequality

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds, where

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^{n} a_i x_i : \left(\sum_{i=1}^{n} |a_i|^{p'} \right)^{\frac{1}{p'}} \leq 1 \right\} \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

The lowest constant M satisfying the above inequality is called the **p-convexity constant of X**, and is denoted by $M^{(p)}(X)$.

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space.

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding,

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E$! $\mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \;!\; \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \;!\; X$

Free Banach lattice over a Banach space

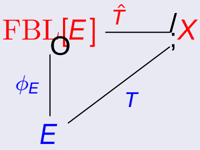
Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \to X$ there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$

Free Banach lattice over a Banach space

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \to X$ there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| = \|T\|$.



Free p-convex Banach lattice over a Banach space

Free p-convex Banach lattice over a Banach space

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \quad p \quad 1$.

Free p-convex Banach lattice over a Banach space

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex Banach lattice over E**

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$,

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$,

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p-convex Banach lattice,

Free p-convex Banach lattice over a Banach space

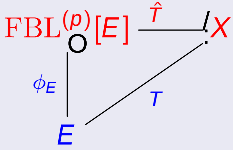
Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \; ! \; \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \; ! \; X$, where X is a p-convex Banach lattice, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \; ! \; X$

Free p-convex Banach lattice over a Banach space

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

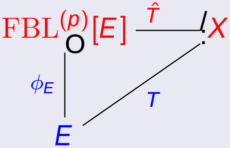
Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p-convex Banach lattice, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| \leq M^{(p)}(X)\|T\|$.



Free p-convex Banach lattice over a Banach space

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

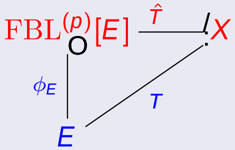
Let E be a Banach space and $1 \le p \le \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p-convex Banach lattice, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| \le M^{(p)}(X)\|T\|$.



Every Banach lattice is 1-convex with constant 1, so $\mathrm{FBL}^{(1)}[E] = \mathrm{FBL}[E]$.

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-convex Banach lattice with p-convexity constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p-convex Banach lattice, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| \leq M^{(p)}(X)\|T\|$.



Observation

$\mathrm{FBL}^{(p)}[E]$ exists and is unique!

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup \left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup\left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p\right)^{\frac{1}{p}} : (x_k^*) \subset E^*,\ \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p\right)^{\frac{1}{p}} \leq 1 \right\}.$$

The set $H_p[E] := \{f \in H[E] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

Let $H[E] := \{f : E \to \mathbb{R} : f(\lambda x) = \lambda f(x)\ \forall x \in E, \lambda \quad 0\}$ be the set of positively homogeneous functions over E. For every $f \in H[E]$ we define

$$\|f\|_p = \sup\left\{ \left(\sum_{k=1}^{n} |f(x_k)|^p\right)^{\frac{1}{p}} : (x_k) \quad E, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k(x)|^p\right)^{\frac{1}{p}} \quad 1 \right\}.$$

The set $H_p[E] := \{f \in H[E] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

We define $\phi_E : E \to H_p[E]$ as $x \mapsto \delta_x(x) = x(x)$, $x \in E$.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E^*] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E^*]$ we define

$$\|f\|_p = \sup\left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

The set $H_p[E^*] := \{f \in H[E^*] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

We define $\phi_E : E \to H_p[E^*]$ as $x \mapsto \delta_x(x^*) = x^*(x)$, $x^* \in E^*$.

The space $\mathrm{FBL}^{(p)}[E] := \overline{\mathrm{lat}}(\phi_E(E)) \subseteq H_p[E^*]$ is a representation of the free p-convex Banach lattice over E.

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

If X is a Banach lattice, X (and thus X) is again a Banach lattice.

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

If X is a Banach lattice, X (and thus X) is again a Banach lattice.

Moreover, if X is p-convex, so is X , and $M^{(p)}(X) = M^{(p)}(X\quad)$.

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

If X is a Banach lattice, X (and thus X) is again a Banach lattice.

Moreover, if X is p-convex, so is X , and $M^{(p)}(X) = M^{(p)}(X \quad)$.

Nevertheless, it remains open whether every dual Banach lattice admits a predual Banach lattice.

Motivation

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

If X is a Banach lattice, X (and thus X) is again a Banach lattice.

Moreover, if X is p-convex, so is X , and $M^{(p)}(X) = M^{(p)}(X\ )$.

Nevertheless, it remains open whether every dual Banach lattice admits a predual Banach lattice.

Given a Banach space E, the aim of this work is to study the interplay between the operations of taking the free (p-convex) Banach lattice and the free dual

What is the relation between **free Banach lattices** and **free duals**?

Duality behaves well with Banach lattices:

If X is a Banach lattice, X (and thus X) is again a Banach lattice.

Moreover, if X is p-convex, so is X , and $M^{(p)}(X) = M^{(p)}(X \ )$.

Nevertheless, it remains open whether every dual Banach lattice admits a predual Banach lattice.

Given a Banach space E, the aim of this work is to study the interplay between the operations of taking the free (p-convex) Banach lattice and the free dual, and to define a free object over E in the category of dual Banach lattices with adjoint lattice homomorphisms.

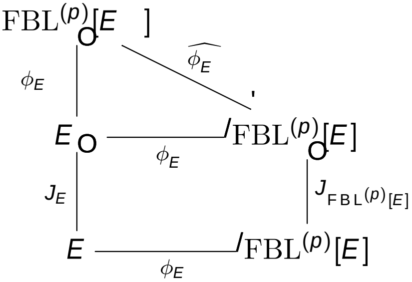
Overview

1. Preliminaries

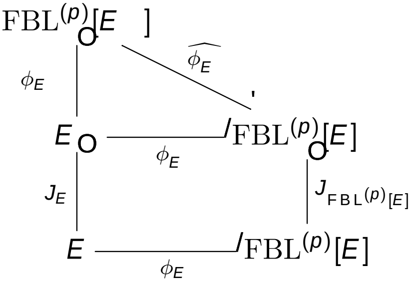
2. $\mathrm{FBL}^{(p)}[E\quad]$ vs $\mathrm{FBL}^{(p)}[E]$

3. Free dual Banach lattices

$\mathrm{FBL}^{(p)}[E\ \ ]$ vs $\mathrm{FBL}^{(p)}[E]$



$\mathrm{FBL}^{(p)}[E\quad]$ vs $\mathrm{FBL}^{(p)}[E]$



Theorem (GS-Tradacete)

The operator $\widehat{\phi_E}$: $\mathrm{FBL}^{(p)}[E\quad]$! $\mathrm{FBL}^{(p)}[E]$ is an isometric lattice embedding.

Key tool

Theorem (Principle of Local Reflexivity)

Let F be a Banach space. For any finite-dimensional subspaces $U \subset F^{**}$ and $V \subset F^{*}$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \subset F$ such that $\|S\|\|S^{-1}\| \leq 1+\epsilon$, $x^{*}(Sx^{**}) = x^{**}(x^{*})$ for every $x^{*} \in V$ and $x^{**} \in U$, and S is the identity on $U \cap J_F(F)$.

Key tool

Theorem (Principle of Local Reflexivity)

Let F be a Banach space. For any finite-dimensional subspaces $U \quad F$ and $V \quad F$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \quad F$ such that $kSkkS \;\; ^{1}k \quad 1+\epsilon$, $x \;\; (Sx \;\;) = x \;\; (x \;\;)$ for every $x \;\; 2\; V$ and $x \;\; 2\; U$, and S is the identity on $U \setminus J_F(F)$.

Using this result, we can show that for every $f \;\, 2 \; \mathrm{FBL}^{(p)}[E \;\;]$

$$\sup \left\{ \left(\sum_{k=1}^{n} jf(x_k \;\;)j^p \right)^{\frac{1}{p}} : (x_k \;\;)_k \quad E \quad , \sup_{x \;\; 2B_E} \left(\sum_{k=1}^{n} jx_k \;\; (x \;\;)j^p \right)^{\frac{1}{p}} \quad 1 \right\} =$$

$$\sup \left\{ \left(\sum_{k=1}^{n} jf \quad J_E(x_k)j^p \right)^{\frac{1}{p}} : (x_k)_k \quad E, \sup_{x \;\; 2B_E} \left(\sum_{k=1}^{n} jx \;\; (x_k)j^p \right)^{\frac{1}{p}} \quad 1 \right\}.$$

Key tool

Theorem (Principle of Local Reflexivity)

Let F be a Banach space. For any finite-dimensional subspaces $U \quad F$ and $V \quad F$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \quad F$ such that $kSk\,kS \;\, ^1k \quad 1+\epsilon$, $x \;\, (Sx \quad) = x \;\, (x \;\,)$ for every $x \;\, 2\, V$ and $x \;\, 2\, U$, and S is the identity on $U \setminus J_F(F)$.

Using this result, we can show that for every $f \;\, 2\, \mathrm{FBL}^{(p)}[E \;\,]$

$$\sup \left\{ \left(\sum_{k=1}^{n} jf\,(x_k \;\,)j^p \right)^{\frac{1}{p}} : (x_k \;\,)_k \quad E \quad , \sup_{x \;\, 2B_E} \left(\sum_{k=1}^{n} jx_k \;\, (x \;\,)j^p \right)^{\frac{1}{p}} \quad 1 \right\} =$$

$$\sup \left\{ \left(\sum_{k=1}^{n} jf \quad J_E(x_k)j^p \right)^{\frac{1}{p}} : (x_k)_k \quad E, \sup_{x \;\, 2B_E} \left(\sum_{k=1}^{n} jx \;\, (x_k)j^p \right)^{\frac{1}{p}} \quad 1 \right\}.$$

This provides an alternative representation for the norm in $\mathrm{FBL}^{(p)}[E \;\,]$ for any dual Banach space $E \;\,$.

$\mathrm{FBL}^{(p)}[E\quad]$ vs $\mathrm{FBL}^{(p)}[E]$

Theorem (GS-Tradacete)

The operator $\widehat{\phi_E\quad}$: $\mathrm{FBL}^{(p)}[E\quad]$! $\mathrm{FBL}^{(p)}[E]$ is an isometric lattice embedding.

$\mathrm{FBL}^{(p)}[E \quad]$ vs $\mathrm{FBL}^{(p)}[E]$

Theorem (GS-Tradacete)

The operator $\widehat{\phi_E} \quad : \mathrm{FBL}^{(p)}[E \quad] \;\; ! \;\; \mathrm{FBL}^{(p)}[E] \quad$ is an isometric lattice embedding.

In other words:

$\mathrm{FBL}^{(p)}[E\ \ ]$ vs $\mathrm{FBL}^{(p)}[E]$

Theorem (GS-Tradacete)

The operator $\widehat{\phi_E}\ \ : \mathrm{FBL}^{(p)}[E\ \ ]\ !\ \mathrm{FBL}^{(p)}[E]$ is an isometric lattice embedding.

In other words:

The free p-convex Banach lattice generated by the free dual over the Banach space E ($\mathrm{FBL}^{(p)}[E\ \ ]$) embeds lattice isometrically into the free dual over the free p-convex Banach lattice generated by E ($\mathrm{FBL}^{(p)}[E]\ \ $).

Overview

1. Preliminaries
2. $\mathrm{FBL}^{(p)}[E\quad]$ vs $\mathrm{FBL}^{(p)}[E]$
3. Free dual Banach lattices

We are now interested in combining the notions of free dual and free (p-convex) Banach lattice over a Banach space E.

Free dual Banach lattices

We are now interested in combining the notions of free dual and free (p-convex) Banach lattice over a Banach space E.

More specifically, we are looking for a free object over a Banach space E in the subcategory $BL^{(p)}$ of p-convex Banach lattices which are duals of some Banach lattice.

Free dual Banach lattices

We are now interested in combining the notions of free dual and free (p-convex) Banach lattice over a Banach space E.

More specifically, we are looking for a free object over a Banach space E in the subcategory $BL^{(p)}$ of p-convex Banach lattices which are duals of some Banach lattice.

Two main questions arise:

Free dual Banach lattices

We are now interested in combining the notions of free dual and free (p-convex) Banach lattice over a Banach space E.

More specifically, we are looking for a free object over a Banach space E in the subcategory $BL^{(p)}$ of p-convex Banach lattices which are duals of some Banach lattice.

Two main questions arise:

- Does such a free object exists for every Banach space?

Free dual Banach lattices

We are now interested in combining the notions of free dual and free (p-convex) Banach lattice over a Banach space E.

More specifically, we are looking for a free object over a Banach space E in the subcategory $BL^{(p)}$ of p-convex Banach lattices which are duals of some Banach lattice.

Two main questions arise:

- Does such a free object exists for every Banach space?
- If so, can we find an explicit construction?

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and 1 p 1 .

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex dual Banach lattice over E**

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$,

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$, along with a linear isometrical embedding $\iota_E : E \to \mathrm{FBL}^{(p)}[E]$,

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$, along with a linear isometrical embedding $\iota_E : E \; ! \; \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded and linear operator $T : E \; ! \; X$,

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}_*[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$, along with a linear isometrical embedding $\iota_E : E \to \mathrm{FBL}^{(p)}_*[E]$, satisfying that for every bounded and linear operator $T : E \to X^*$, X^* being the dual of a Banach lattice X and p-convex,

Free p-convex dual Banach lattice

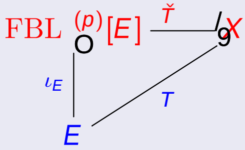
Definition

Let E be a Banach space and $1 \quad p \quad 1$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$, along with a linear isometrical embedding $\iota_E : E \; ! \; \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded and linear operator $T : E \; ! \; X$, X being the dual of a Banach lattice X and p-convex, there exists a unique weak* to weak* continuous lattice homomorphism $\check{T} : \mathrm{FBL}^{(p)}[E] \; ! \; X$,

Free p-convex dual Banach lattice

Definition

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex dual Banach lattice over E** is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant 1, which is the dual of some other Banach lattice $Z_E^{(p)}$, along with a linear isometrical embedding $\iota_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded and linear operator $T : E \to X$, X being the dual of a Banach lattice X and p-convex, there exists a unique weak* to weak* continuous lattice homomorphism $\check{T} : \mathrm{FBL}^{(p)}[E] \to X$, such that $\check{T} \circ \iota_E = T$, and moreover, $\|\check{T}\| \leq M^{(p)}(X)\|T\|$.



Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X p-convex, X Banach lattice

Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X p-convex, X Banach lattice) X p -concave, p $<$ *1*

Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X p-convex, X Banach lattice) X p -concave, $p < 1$) X order continuous

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X p-convex, X Banach lattice) X p -concave, p $<$ 1) X order continuous) J_X interval preserving

Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}\ ^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X *p*-convex, *X* Banach lattice *)* *X* *p* -concave, *p* $<$ *1* *)* *X* order continuous *)* J_X interval preserving *)* J_X lattice homomorphism.

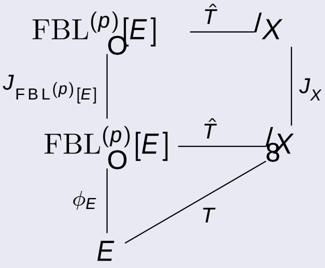
Case $p > 1$

Theorem (GS-Tradacete)

The space $\mathrm{FBL}^{(p)}[E]$ satisfies the definition of $\mathrm{FBL}^{(p)}[E]$ for every $p > 1$.

Sketch of the proof.

X p-convex, X Banach lattice) X p-concave, $p < 1$) X order continuous) J_X interval preserving) J_X lattice homomorphism.



Case $p = 1$

The previous argument fails for $p = 1$, since we need to be able to extend to the duals of Banach lattices which are not order continuous, such as $C[0, 1]$.

Case $p = 1$

The previous argument fails for $p = 1$, since we need to be able to extend to the duals of Banach lattices which are not order continuous, such as $C[0, 1]$.

In fact, $\mathrm{FBL}[E]$ fails to be the free dual Banach lattice over E as long as E contains a complemented copy of ℓ_1.

Theorem (GS-Tradacete)

Let E be a Banach space. The following are equivalent:

1. E does not contain a complemented copy of ℓ_1.
2. The space $\mathrm{FBL}[E]$ satisfies the definition of $\mathrm{FBL}\ [E]$.

Case $p = 1$

The previous argument fails for $p = 1$, since we need to be able to extend to the duals of Banach lattices which are not order continuous, such as $C[0,1]$.

In fact, $\mathrm{FBL}[E]$ fails to be the free dual Banach lattice over E as long as E contains a complemented copy of ℓ_1.

Theorem (GS-Tradacete)

Let E be a Banach space. The following are equivalent:

1. E does not contain a complemented copy of ℓ_1.
2. The space $\mathrm{FBL}[E]$ satisfies the definition of $\mathrm{FBL}\ [E]$.

We do not know yet if $\mathrm{FBL}\ [E]$ exists when E contains a complemented copy of ℓ_1.

Thank you for your attention!

Main references

- A. Avilés, J. Rodríguez, P. Tradacete, *The free Banach lattice generated by a Banach space*. J. Funct. Anal. **274** (10) (2018), 2955–2977.
- M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*. Springer, New York, 2011.
- N. Ghoussoub, W. B. Johnson, *Factoring operators through Banach lattices not containing $C(0,1)$*. Math. Z. **194** (1987), 153–171.
- H. Jardón-Sánchez, N. J. Laustsen, M. A. Taylor, P. Tradacete, V. G. Troitsky, *Free Banach lattices under convexity conditions*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (15) (2022).
- T. Oikhberg, M. A. Taylor, P. Tradacete, V. G. Troitsky, *Free Banach lattices*. Preprint (2022).