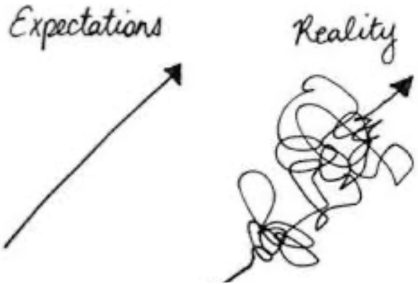
Conditional indicators

Dorsaf Cherif
Carthage University
Joint-work with Emmanuel Lepinette
Paris Dauphine University

University of Ljubljana, July 2023

Joke

Why does a random variable go to therapy?
Because "she" has too many unrealistic conditional expectations.



What is a conditional indicator and what for ?

Settings

We consider

- $\mathcal{F}$ a complete $\sigma-$ algebra

Settings

We consider

- $\mathcal{F}$ a complete $\sigma-$ algebra
- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

Settings

We consider

- $\mathcal{F}$ a complete $\sigma-$ algebra
- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.
- $\mathbb{L}^0(\mathbf{R}, \mathcal{F})$ the set of $\mathcal{F}$-measurable functions taking values in $\mathbf{R}$.

Settings

We consider

- $\mathcal{F}$ a complete $\sigma-$ algebra
- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.
- $\mathbb{L}^0(\mathbf{R}, \mathcal{F})$ the set of $\mathcal{F}$-measurable functions taking values in $\mathbf{R}$.
- $\mathcal{H}$ a sub σ-algebra of $\mathcal{F}$.

The most famous example: The conditional expectation

Let $X \in \mathbb{L}^1(\mathbf{R}, \mathcal{F})$. There exists a unique almost surely random variable in $\mathbb{L}^1(\mathbf{R}, \mathcal{H})$, denoted by $\mathbb{E}[X|\mathcal{H}]$ and called **the conditional expectation**, such that

$$\forall B \in \mathcal{H}, \quad \mathbb{E}[X 1_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}] 1_B].$$

The most famous example: The conditional expectation

Let $X \in \mathbb{L}^1(\mathbf{R}, \mathcal{F})$. There exists a unique almost surely random variable in $\mathbb{L}^1(\mathbf{R}, \mathcal{H})$, denoted by $\mathbb{E}[X|\mathcal{H}]$ and called **the conditional expectation**, such that

$$\forall B \in \mathcal{H}, \quad \mathbb{E}[X 1_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}] 1_B].$$

If $\mathcal{H}_0 = \{\emptyset, \Omega\}$ then $E[.|\mathcal{H}_0] = E[.]$.

Conditional supremum

Another less famous typical example : The conditional supremum.

The **conditional essentiel supremum** of a random variable $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ given $\mathcal{H}$ is the unique $\mathcal{H}$ measurable random variable denoted by ess $\sup_{\mathcal{H}}(X)$ which satisfies

$$\text{ess}\sup_{\mathcal{H}}(X) = \inf\{Y \in \mathbb{L}^0(\mathcal{H}, \mathbb{R}) \text{ such that } X \leq Y\}.$$

Conditional supremum

Dorsaf Cherif
Carthage University
Joint-work with
Emmanuel Lepinette
Paris Dauphine University

Another less famous typical example : The conditional supremum.

The **conditional essentiel supremum** of a random variable $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ given $\mathcal{H}$ is the unique $\mathcal{H}$ measurable random variable denoted by $\text{ess sup}_{\mathcal{H}}(X)$ which satisfies

$$\text{ess sup}_{\mathcal{H}}(X) = \inf\{Y \in \mathbb{L}^0(\mathcal{H}, \mathbb{R}) \text{ such that } X \leq Y\}.$$

Analogously we define **the conditional essentiel infimum** as ,

$$\text{ess inf}_{\mathcal{H}}(X) = -\,\text{ess sup}_{\mathcal{H}}(-X).$$

Conditional supremum

Another less famous typical example : The conditional supremum.

The **conditional essentiel supremum** of a random variable $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ given $\mathcal{H}$ is the unique $\mathcal{H}$ measurable random variable denoted by $\text{ess sup}_{\mathcal{H}}(X)$ which satisfies

$$\text{ess sup}_{\mathcal{H}}(X) = \inf\{Y \in \mathbb{L}^0(\mathcal{H}, \mathbb{R}) \text{ such that } X \le Y\}.$$

Analogously we define **the conditional essentiel infimum** as ,

$$\text{ess inf}_{\mathcal{H}}(X) = -\text{ ess sup}_{\mathcal{H}}(-X).$$

If $\mathcal{H}_0 = \{\emptyset, \Omega\}$ then $\text{ess sup}_{\mathcal{H}} = \text{ess sup}$.

Definition of a conditional indicator

We introduce a conditional indicator in our work [1] in the following way:

Definition

Let $\mathbb{D}_I$ be a subset of $\mathbb{L}^0(\mathbf{R}, \mathcal{F})$ containing 0. We say that a mapping

$$\begin{array}{rcl} I_{\mathcal{H}} : \mathbb{D}_I & \longrightarrow & \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}). \\ X & \longmapsto & I_{\mathcal{H}}(X) \end{array}$$

is a Conditional Indicator (C.I.) if the following properties hold:

(P1) $\text{ess inf}_{\mathcal{H}}(X) \leq I_{\mathcal{H}}(X) \leq \text{ess sup}_{\mathcal{H}}(X)$ a.s.

(P2) $\mathbb{D}_I + \mathbb{L}^0(\mathbf{R}, \mathcal{H}) \subseteq \mathbb{D}_I$.

Properties

Let $I_{\mathcal{H}}$ be a C.I. w.r.t a sub-σ-algebra $\mathcal{H}$. Then,

1) $I_{\mathcal{H}}$ is said $\mathcal{H}$-translation invariant if for all $X \in D_I$ and $Y_{\mathcal{H}} \in L^0(\mathbf{R}, \mathcal{H})$ we have

$$I_{\mathcal{H}}(X + Y_{\mathcal{H}}) = I_{\mathcal{H}}(X) + Y_{\mathcal{H}}.$$

Properties

Let $I_{\mathcal{H}}$ be a C.I. w.r.t a sub-σ-algebra $\mathcal{H}$. Then,

1) $I_{\mathcal{H}}$ is said $\mathcal{H}$-translation invariant if for all $X \in D_I$ and $Y_{\mathcal{H}} \in L^0(\mathbf{R}, \mathcal{H})$ we have

$$I_{\mathcal{H}}(X + Y_{\mathcal{H}}) = I_{\mathcal{H}}(X) + Y_{\mathcal{H}}.$$

2) $I_{\mathcal{H}}$ is said $\mathcal{H}$-positively-homogeneous if, for every $\alpha_{\mathcal{H}} \in L^0(\mathbf{R}^+, \mathcal{H})$, we have $\alpha_{\mathcal{H}} D_I \subset D_I$ and for any $X \in D_I$,

$$I_{\mathcal{H}}(\alpha_{\mathcal{H}} X) = \alpha_{\mathcal{H}} I_{\mathcal{H}}(X).$$

Properties

Let $I_{\mathcal{H}}$ be a C.I. w.r.t a sub-σ-algebra $\mathcal{H}$. Then,

1) $I_{\mathcal{H}}$ is said $\mathcal{H}$-translation invariant if for all $X \in D_I$ and $Y_{\mathcal{H}} \in L^0(\mathbf{R}, \mathcal{H})$ we have

$$I_{\mathcal{H}}(X + Y_{\mathcal{H}}) = I_{\mathcal{H}}(X) + Y_{\mathcal{H}}.$$

2) $I_{\mathcal{H}}$ is said $\mathcal{H}$-positively-homogeneous if, for every $\alpha_{\mathcal{H}} \in L^0(\mathbf{R}^+, \mathcal{H})$, we have $\alpha_{\mathcal{H}} D_I \subset D_I$ and for any $X \in D_I$,

$$I_{\mathcal{H}}(\alpha_{\mathcal{H}} X) = \alpha_{\mathcal{H}} I_{\mathcal{H}}(X).$$

3) $I_{\mathcal{H}}$ is said $\mathcal{H}$-linear if, for all $\alpha_{\mathcal{H}} \in L^0(\mathbf{R}, \mathcal{H})$, $\alpha_{\mathcal{H}} D_I + D_I \subset D_I$, and for every $X, Y \in D_I$,

$$I_{\mathcal{H}}(\alpha_{\mathcal{H}} X + Y) = \alpha_{\mathcal{H}} I_{\mathcal{H}}(X) + I_{\mathcal{H}}(Y).$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete filtration, i.e. a sequence of complete σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$.

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete filtration, i.e. a sequence of complete σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$.

Consider a family $(I_t)_{t\in[0,T]}$ of adapted conditional indicators in the sense that I_t is a conditional indicator w.r.t. $\mathcal{F}_t$, for every $t \in [0, T]$. We say that $(I_t)_{t\in[0,T]}$ satisfies:

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete filtration, i.e. a sequence of complete σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$.

Consider a family $(I_t)_{t\in[0,T]}$ of adapted conditional indicators in the sense that I_t is a conditional indicator w.r.t. $\mathcal{F}_t$, for every $t \in [0, T]$. We say that $(I_t)_{t\in[0,T]}$ satisfies:

1) the tower property if, for any $s \leq t$, $I_t(\mathbb{D}_{I_s}) \subseteq \mathbb{D}_{I_s} \subseteq \mathbb{D}_{I_t}$ and

$$I_s(I_t(X)) = I_s(X), \text{ for all } X \in \mathbb{D}_{I_s}.$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete filtration, i.e. a sequence of complete σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$.

Consider a family $(I_t)_{t\in[0,T]}$ of adapted conditional indicators in the sense that I_t is a conditional indicator w.r.t. $\mathcal{F}_t$, for every $t \in [0, T]$. We say that $(I_t)_{t\in[0,T]}$ satisfies:

1) the tower property if, for any $s \leq t$, $I_t(\mathbb{D}_{I_s}) \subseteq \mathbb{D}_{I_s} \subseteq \mathbb{D}_{I_t}$ and

$$I_s(I_t(X)) = I_s(X), \text{ for all } X \in \mathbb{D}_{I_s}.$$

2) the projection property if $\mathbb{D}_{I_0}$ is $\mathcal{F}_t$-decomposable for every $t \geq 0$ and the following condition holds:

$$\mathrm{Pr} : I_0(X1_{F_t}) = I_0(I_t(X)1_{F_t}), \text{ for all } F_t \in \mathcal{F}_t.$$

Questions

Important questions arise here:

1. What do we need for a conditional indicator to be linear?
2. What are the optimal conditions for a CI to be a conditional expectation ?
3. Does the conditional supremum satisfy the projection property ?
4. Does the stopping theorem work for the conditional supremum?

Linearity

Theorem

Let $I_{\mathcal{H}}$ be a C.I. w.r.t. a sub-σ-algebra $\mathcal{H}$, defined on a vector sub-space $\mathbb{D}_I$ of $L^0(\mathbf{R}, \mathcal{F})$. Then,

$$I_{\mathcal{H}} \text{ is additive} \iff I_{\mathcal{H}} \text{ is } \mathcal{H}\text{-linear.}$$

History

- Moy, in 1954 , has characterized in [5] conditional expectations as additive and positively homogeneous maps on $(L^0)^+$, satisfying the monotone convergence and the averaging propertie, mapping bounded functions into bounded functions.

History

- Moy, in 1954 , has characterized in [5] conditional expectations as additive and positively homogeneous maps on $(L^0)^+$, satisfying the monotone convergence and the averaging propertie, mapping bounded functions into bounded functions.
- Douglas, in 1965, has shown in [4] that the conditional expectations are the only contractive projections on L^1 that leaves the constant functions invariant. Ando generalized this characterization to L^p spaces.

History

- Moy, in 1954 , has characterized in [5] conditional expectations as additive and positively homogeneous maps on $(L^0)^+$, satisfying the monotone convergence and the averaging propertie, mapping bounded functions into bounded functions.
- Douglas, in 1965, has shown in [4] that the conditional expectations are the only contractive projections on L^1 that leaves the constant functions invariant. Ando generalized this characterization to L^p spaces.
- De Pagter, Dodds and Huijsmans, in 1990, have proved in [6] that each strictly positive order continuous projection for which the constant functions are invariant is of conditional expectation type.

Characterization of a CEO

Theorem

Suppose that $I_{\mathcal{H}}$ is a conditional indicator with reference to a sub-σ-algebra $\mathcal{H}$ defined on $\mathbb{L}^1(\mathbf{R},\mathcal{F})$ and satisfies the following properties:

1) $I_{\mathcal{H}}(X+Y) = I_{\mathcal{H}}(X) + I_{\mathcal{H}}(Y)$ *for all* $X, Y \in \mathbb{L}^1(\mathbf{R},\mathcal{F})$.
2) *the Fatou property i.e., for any sequence* $(X_n)_n$ *of* $\mathbb{L}^1(\mathbf{R},\mathcal{F})$, *we have* $I_{\mathcal{H}}(\liminf_n X_n) \leq \liminf I_{\mathcal{H}}(X_n)$.

Then, there exists a probability measure $\mu << \mathbb{P}$, *with* $\rho = d\mu/d\mathbb{P} \in \mathbb{L}^1(\mathbf{R}_+,\mathcal{F})$ *such that , for all* $X \in \mathbb{L}^1(\mathbf{R},\mathcal{H})$,

$$I_{\mathcal{H}}(X) = E_\mu(X|\mathcal{H}) = \mathbb{E}(\rho X|\mathcal{H})$$

Projection property for the conditional supremum (CS)

Consider $\mathcal{H}_0 = \{\emptyset, \Omega\}$. Then $\text{ess sup}_{\mathcal{H}_0} = \text{ess sup}$,
By the tower property we obtain

$$\text{ess sup}(\text{ess sup}_{\mathcal{H}}(X)1_A) = \text{ess sup}(X1_A), \forall A \in \mathcal{H}.$$

We recall that:

$$\forall B \in \mathcal{H}, \quad \mathbb{E}[\mathbb{E}[X|\mathcal{H}]1_B] = \mathbb{E}[X1_B].$$

Projection property for the conditional supremum (CS)

Consider $\mathcal{H}_0 = \{\emptyset, \Omega\}$. Then $\text{ess sup}_{\mathcal{H}_0} = \text{ess sup}$,
By the tower property we obtain

$$\text{ess sup}(\text{ess sup}_{\mathcal{H}}(X)1_A) = \text{ess sup}(X1_A), \forall A \in \mathcal{H}.$$

We recall that:

$$\forall B \in \mathcal{H}, \quad \mathbb{E}[\mathbb{E}[X|\mathcal{H}]1_B] = \mathbb{E}[X1_B].$$

Here we aim to see under which conditions we have the

uniqueness.

Theorem

Let $\mathcal{H}_0 = \{\emptyset, \Omega\}$ *and* $\mathcal{H}$ *a sub-*σ*-algebras of* $\mathcal{F}$*. Let* $X \in \mathbb{L}^0(\mathbf{R}^+, \mathcal{F})$ *such that* $\text{ess sup}_{\mathcal{H}_0}(X) \in \mathbf{R}$*. Then there exists a unique* $Z \in \mathbb{L}^0(\mathbf{R}^+, \mathcal{H})$ *such that* $\text{ess sup}_{\mathcal{H}_0}(Z) \in \mathbf{R}$*, satisfying*

$$\text{ess sup}_{\mathcal{H}_0}(Z1_A) = \text{ess sup}_{\mathcal{H}_0}(X1_A), \forall A \in \mathcal{H}.$$

More precisely $Z = \text{ess sup}_{\mathcal{H}}(X)$

Consequence

Proposition

Let $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$ such that $X > 0$ and $\operatorname{ess\,sup}_{\mathcal{H}_0}(X) \in \mathbf{R}$. Then there exists a unique $Z \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$ such that $\operatorname{ess\,sup}_{\mathcal{H}_0}(Z) \in \mathbf{R}$, satisfying

$$\operatorname{ess\,sup}_{\mathcal{H}_0}(Z1_A) = \operatorname{ess\,sup}_{\mathcal{H}_0}(X1_A), \forall A \in \mathcal{H}.$$

Counter Examples :

- Take $\Omega = \mathbf{R}$, $\mathcal{F} = \mathcal{B}(\mathbf{R})$ and $\mathbb{P}$ the probability mesure such that $d\mathbb{P} = 1/(1+x^2)d\mu$ with μ the Lebesgue measure. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- Let $X = 0$ almost surely and $Z(\omega) = -exp(\omega)$ for all $\omega \in \Omega$.
- Let $A = \{Z \leq -1\}$ and $\mathcal{F}_1 = \sigma(1_A)$. So $\mathcal{F}_1 = \{A, A^c, \Omega, \emptyset\}$.
- Now consider $Z_1 = \text{ess sup}_{\mathcal{F}_1}(Z)$. Then $Z_1 = -1$ on A and $Z_1 = 0$ on A^c. So $Z_1 \neq 0 = \text{ess sup}_{\mathcal{F}_1}(X)$. However

$$\text{ess sup}_{\mathcal{F}_0}(Z_1 1_B) = 0 = \text{ess sup}_{\mathcal{F}_0}(X 1_B) \quad \forall B \in \mathcal{F}_1.$$

Martingales for the CS indicator

Definition

Let $(M_t)_t$ be an adapted process to the filtration $(\mathcal{F}_t)_t$. We say that $(M_t)_t$ is a martingale for the CS indicator if for any $s \leq t \leq T$,

$$\text{ess sup}_{\mathcal{F}_s}(M_t) = M_s.$$

We call it a supermartingale if for any $s \leq t \leq T$,

$$\text{ess sup}_{\mathcal{F}_s}(M_t) \geq M_s$$

and a submartingale

$$\text{ess sup}_{\mathcal{F}_s}(M_t) \leq M_s.$$

Stopping theorem for the CS indicator

We recall that for a martingale $(X_n)_{n\geq 0}$ and two bounded stopping times τ and S such that $\tau \leq S$ almost surely, we obtain

$$\mathbb{E}[X_S|\mathcal{F}_\tau] = X_\tau.$$

Stopping theorem for the CS indicator

We recall that for a martingale $(X_n)_{n\geq 0}$ and two bounded stopping times τ and S such that $\tau \leq S$ almost surely, we obtain

$$\mathbb{E}[X_S|\mathcal{F}_\tau] = X_\tau.$$

Theorem

Let $(M_t)_t$ be a martingales for the CS indicator . Let τ , s be two stopping times such that $s \leq \tau \leq T$. Then

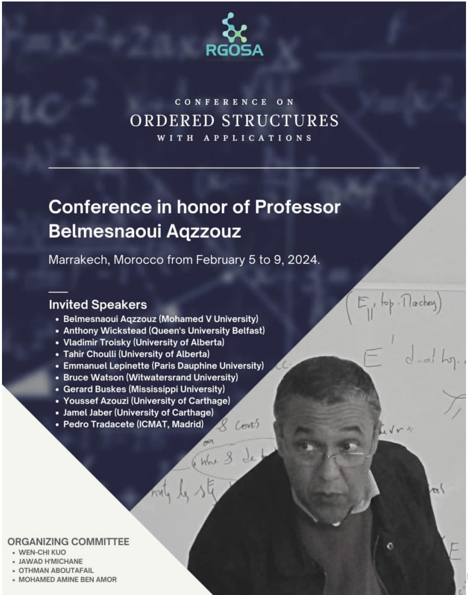
$$\text{ess sup}_{\mathcal{F}_s}(M_\tau) = M_s.$$

(for supermartingale or submartingale we replace the equality by inequalities).

References

- Conditional indicators. Cherif Dorsaf, Emmanuel Lepinette.
- No-arbitrage conditions and pricing from discrete-time to continuous-time strategies. D.Cherif, E.Lepinette.
- Kabanov, Y. and Safarian, M., Markets with transaction costs. Mathematical Theory. Springer-Verlag, 2009.
- Contractive projections on an L1 space. R. G. DOUGLAS ,Pacific Journal of Mathematics Vol. 15, No. 2, 1965
- Shu-Teh Chen Moy, Characterizations of conditional expectations as a transformation on function spaces, Pacific J. Math., 4 (1954), 47-64.
- Characterizations of conditional expctation-type operators. Peter Gerard Dodds, C.B.Huijsmans and Bernardus De Pagter. November 1990.

Thank you for your attention.


RGOSA
CONFERENCE ON
ORDERED STRUCTURES
WITH APPLICATIONS
Conference in honor of Professor Belmesnaoui Aqzzouz
Marrakech, Morocco from February 5 to 9, 2024.
Invited Speakers
• Belmesnaoui Aqzzouz (Mohamed V University)
• Anthony Wickstead (Queen's University Belfast)
• Vladimir Troisky (University of Alberta)
• Tahir Choulli (University of Alberta)
• Emmanuel Lepinette (Paris Dauphine University)
• Bruce Watson (Witwatersrand University)
• Gerard Buskes (Mississippi University)
• Youssef Azouzi (University of Carthage)
• Jamel Jaber (University of Carthage)
• Pedro Tradacete (ICMAT, Madrid)
ORGANIZING COMMITTEE
• WEN-CHI KUO
• JAWAD H'MICHANE
• OTHMAN ABOUTAFAIL
• MOHAMED AMINE BEN AMOR