Hilbert geometries isometric to Banach spaces

Cormac Walsh

Inria and Ecole Polytechnique, Paris, France

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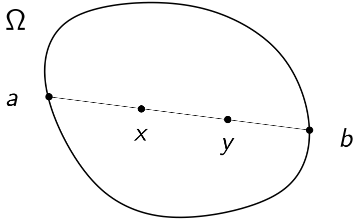
Hilbert's metric

In a letter to Klein in 1894, Hilbert generalised Klein's model of hyperbolic space.



Definition

Hilbert metric

$$d_F(x, y) := \frac{1}{2} \log \frac{|ay||bx|}{|ax||by|}.$$

If Ω is a disk, then Ω is isometric to the hyperbolic plane.

Bonsall's $M(\cdot,\cdot)$ function

Let (V, C, u) be an order-unit space.

Definition

$$M(x,y) := \inf\{\lambda > 0 \mid x \leq \lambda y\}, \qquad \text{for } x, y \in \operatorname{int} C.$$

Proposition

$$M(x,y) = \sup_{z \in C^*} \frac{\langle z, x\rangle}{\langle z, y\rangle}$$

Example

For the positive cone $\mathbb{R}^n_+$,

$$M(x,y) = \max_i \frac{x_i}{y_i}.$$

The Hilbert pseudo-metric on the cone

The Hilbert (pseudo-)metric is defined to be, for $x, y \in \operatorname{int} C$,

$$\tilde{d}_H(x, y) := \frac{1}{2} \log M(x, y) M(y, x).$$

Hilbert's metric satisfies

- (positivity) $\tilde{d}_H(x, y) \geq 0$;
- (pseudo-definiteness) $\tilde{d}_H(x, y) = 0$ iff $x = \lambda y$ for some $\lambda > 0$;
- (symmetry) $\tilde{d}_H(x, y) = \tilde{d}_H(y, x)$;
- (triangle inequality) $\tilde{d}_H(x, z) \leq \tilde{d}_H(x, y) + \tilde{d}_H(y, z)$.

Proposition For x and y in a cross-section Ω of the cone,

$$d_H(x, y) = \tilde{d}_H(x, y).$$

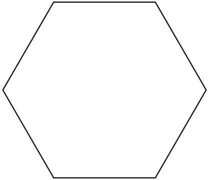
The case of simplices

Proposition (Nussbaum, de la Harpe)

Ω is an n-simplex $\Longrightarrow (\Omega, d_H)$ *is isometric to a normed space*

The unit ball of the normed space:

dim $= 2$



hexagon

dim $= 3$

rhombic dodecahedron

Theorem (Foertsch–Karlsson)

Ω is an n-simplex $\Longleftrightarrow (\Omega, d_H)$ *is isometric to a finite-dimensional normed space*

Infinite-dimensional "simplices"

Definition

- $\mathcal{C}(K)$ the continuous functions on a compact Hausdorff space K;
- $\mathcal{C}^+(K)$ the positive continuous real-valued functions on K;
- $\operatorname{cl}\mathcal{C}^+(K)$ the non-negative continuous functions on K;
- u the function on K that is identically 1.

$(\mathcal{C}(K), \operatorname{cl}\mathcal{C}^+(K), u)$ is an order-unit space.

$$d_H(x,y) = \frac{1}{2}\log \sup_{j,k\in K} \frac{x(j)\,y(k)}{y(j)\,x(k)}, \qquad \text{for } x,y \in \mathcal{C}^+(K).$$

This is isometric to $(C(K), ||\cdot||_H)$, where

$$||x||_H := \frac{1}{2}\sup_{j\in K} x(j) - \frac{1}{2}\inf_{j\in K} x(j).$$

Generalisation of the Foertsch–Karlsson result to infinite dimension

Theorem (W.)

The Hilbert geometry on a cone C of an order-unit space is isometric to a Banach space $\iff$ C is linearly isomorphic to $\operatorname{cl}\mathcal{C}^+(K)$*, for some compact Hausdorff space K.*

Main tool in proof: the Horofunction Boundary

Let (X, d) be a metric space.

Denote by $C(X)$ the space of continuous real-valued functions on X, with the topology of uniform convergence on compact sets.

Definition (Horofunction boundary [Gromov 1978])

Choose a base point b.
To each $x \in X$, associate the function

$$\phi_x(\cdot) := d(\cdot, x) - d(b, x).$$

The map

$$\Phi \colon X \to C(X) \colon x \mapsto \phi_x$$

is a continuous injection.
The *horofunction compactification* is $\operatorname{cl} \operatorname{Im} X$. The *horofunction boundary* is $\operatorname{cl} \operatorname{Im} X \setminus \operatorname{Im} X$. Its elements are called *horofunctions*.

Busemann points in the boundary

Let γ be a function $\mathbb{R}_+ \to X$.

Definition

γ is a *almost-geodesic ray* if, for every $\epsilon > 0$,

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon$$

for $s, t \in T$ large enough with $s \leq t$.

Proposition

Every almost-geodesic converges to a point in the compactifiaction.

Definition

A horofunction is a *Busemann point* if there exists an almost-geodesic converging to it.

The detour metric

[Akian–Gaubert-W.]

For two horofunctions ξ and η, define the *detour cost*:

$$H(\xi,\eta) = \liminf_{x\to\xi} \lim_{y\to\eta} \big(d(b,x) + d(x,y) - d(b,y)\big),$$

Define the *detour metric*:

$$\delta(\xi,\eta) := H(\xi,\eta) + H(\eta,\xi).$$

Proposition

δ is a metric on the set of Busemann points (but might take the value $+\infty$).

We can partition the set of Busemann points so that δ is finite when restricted to each subset.

We call these subsets the *parts* of the horofunction boundary. δ is a genuine metric on each part.

Singleton horofunctions

Definition

A Busemann point ξ is a *singleton* if $\delta(\xi, \eta) = \infty$, for all other Busemann points η.

Lemma (Singletons of a Banach space)

The singletons of a Banach space are exactly the extreme points of the dual ball.

Singletons of a Hilbert geometry

Idea — split the Hilbert metric into two pieces:

$$d(x, y) = \frac{1}{2} \log M(x, y) + \frac{1}{2} \log M(y, x)$$

Funk metric:

$$d_F(x, y) := \log M(x, y)$$

Every Hilbert horofunction is the sum (divided by 2) of a Funk horofunction and a reverse-Funk horofunction.

Work out the singletons of the Funk geometry and the reverse-Funk geometry seperately, and combine the results.

Lemma

Every singleton s of a Hilbert geometry can be written $s_H = (1/2)(s_F + s_R)$, where s_F and s_R are singletons of the Funk and reverse-Funk geometries, respecively.

Singletons of Hilbert geometries that are isometric to Banach spaces

Assume there is an isometry $\Phi\colon X \to P(C)$.

Observation: If s is a singleton of the Banach space X, then so is $-s$.

Lemma

If s is a singleton of the Banach space, and

$$s = \frac{1}{2}(s_F + s_R) \circ \Phi \qquad \text{and} \qquad -s = \frac{1}{2}(s'_F + s'_R) \circ \Phi,$$

then $s_F = -s_R$ and $s_R = -s_F$.

Lemma

If a Hilbert geometry is isometric to a Banach space, then the singleton Busemann points of the Hilbert geometry are exactly the functions of the form $(1/2)(f_1 - f_2)$, with f_1 and f_2 distinct Funk singletons.

Sketch of proof of main theorem

Let $\Phi\colon X \to P(C)$ be the isometry between the Banach space and Hilbert geometry.

Add a dimension $X' := X \times \mathbb{R}$.

Want to show that X' looks like $C(K)$ with the seminorm $||\cdot||_H$. Take $K := \text{cl}\{\text{Funk singletons}\}$.

Extend Φ to a map $\Phi'\colon X' \to C$ such that the pullback satisfies $f_0 \circ \Phi'(x, \alpha) = \alpha$, for some Funk singleton f_0.

For all Funk singletons f, we have $f - f_0$ is a Hilbert singleton
$\implies (f - f_0) \circ \Phi'$ is a Banach space singleton, and hence linear
$\implies f \circ \Phi'$ is linear.

So, each element of K gives a linear functional on X'.

Sketch of proof of main theorem, continued

In the Banach space $(X, ||\cdot||)$, we have

$$||x|| = \sup_{s \in S_X} s(x).$$

$(S_X := \{\text{singletons of } X\})$
Extend each $s \in S_X$ to X': let $s(x,\alpha) := s(x)$. Extend $||\cdot||$ to X' by ignoring the second coordinate.

So,

$$\begin{aligned} ||(x,\alpha)|| = ||x|| &= \sup_{s \in S_X} s(x) \\ &= \sup_{f_1, f_2 \in S_F} (1/2)\big(f_1 - f_2\big) \circ \Phi \\ &= \frac{1}{2} \sup_{f_1 \in S_F} f_1 \circ \Phi - \frac{1}{2} \inf_{f_2 \in S_F} f_2 \circ \Phi \\ &= ||x||_H. \end{aligned}$$