Linear operators on Banach function spaces on groups which commute with translations

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POSI+IVITY XI, Ljubljana, July 13, 2023

Outline

- Review of Brainerd and Edwards's results on linear operators that commute with translations.
- The relationship between density of $C_c(G)$ in a translation invariant Banach function space E on a locally compact group G and order continuity of the norm of E.
- Generalization of the Brainerd and Edwards's result on L^p spaces to translation invariant Banach function spaces on locally compact groups.

Notations

- G: a locally compact group with a left Haar measure
- $C_c(G)$: the space of compactly supported continuous functions on G
- $C_c(G)^\sim$: the order dual of $C_c(G)$
- (right) translation $R_x (x \in G)$ for a function f on G:

$$R_x f(y) = f(yx) \quad (y \in G)$$

- (right) translation $R_x (x \in G)$ for $\mu \in C_c(G)^\sim$:

$$\langle f, R_x \mu \rangle = \langle R_{x^{-1}} f, \mu \rangle \quad (f \in C_c(G))$$

Convolutions

Let f be a locally integrable function on G and $\mu \in C_c(G)^\sim$. The *convolution* $\mu * f$ of μ with f is defined as the $C_c(G)^\sim$-valued integral

$$\mu * f := \int_G f(x^{-1})(R_x\mu)\,\mathrm{d}x$$

if it exists with respect to the weak* topology on $C_c(G)^\sim$, i.e., $\mu * f$ is a unique element in $C_c(G)^\sim$ such that
$\langle h, \mu * f\rangle = \int_G f(x^{-1})\langle h, R_x\mu\rangle\,\mathrm{d}x \quad (h \in C_c(G))$.
When μ is identified with a function $g \in C_c(G)$, $g * f$ coincides with the pointwise defined function on G:

$$x \mapsto \int_G f(y)g(xy)\,\mathrm{d}y.$$

Brainerd and Edwards's results

Theorem

Suppose $T\colon C_c(G) \to C_c(G)^\sim$ is a positive operator that commutes with right translations. Then T is of the form $f \mapsto \mu * f$ for some positive element $\mu \in C_c(G)^\sim$.

Let $\mathcal{C}_R(G)$ denote the set of regular operators from $C_c(G)$ to $C_c(G)^\sim$ that commute with right translations. The above theorem gives a one-to-one correspondence between $C_c(G)^\sim$ and $\mathcal{C}_R(G)$.

Theorem

There exists an order isomorphism between $C_c(G)^\sim$ and $\mathcal{C}_R(G)$. Furthermore, the isomorphism is also a topological isomorphism if $C_c(G)^\sim$ and $\mathcal{C}_R(G)$ are equipped with the following topologies:
On $C_c(G)^\sim$: the coarsest topology such that all functionals of the form $\mu \mapsto \langle |f|, \mu\rangle$ for $f \in C_c(G)$ are continuous.
On $\mathcal{C}_R(G)$: the coarsest topology such that all functionals of the form $T \mapsto \langle g, |T|f\rangle$ for $f, g \in C_c(G)$ are continuous.

Brainerd and Edwards's results

Theorem

If $T : L^p(G) \to C_c(G)^\sim$ is a positive operator that commutes with right translations, then T is of the form $f \mapsto \mu * f$ for some positive $\mu \in C_c(G)^\sim$ for $1 \leq p < \infty$.

The differences between $L^p(G)$ spaces when p is finite and when p is infinite:

(a) the L^p norm is order continuous when $1 \leq p < \infty$ but generally not when $p = \infty$.

(b) $L^p(G)$ is contains $C_c(G)$ as a dense subspace when $1 \leq p < \infty$ but generally not when $p = \infty$.

Density of $C_c(G)$ and order continuity of the norm

A *Banach function space* on G is an order ideal of the space $L^0(G)$ of all measurable functions (modulo equality almost everwhere) equipped with a Banach lattice norm.

Theorem

Suppose E is a translation invariant Banach function space on G, and that the maps $x \mapsto \|R_x\|$ are bounded on compact subsets of G. The density of $C_c(G)$ in E implies that the norm on E is order continuous.
The reverse implication holds if every element of E has a σ-finite support.

Density of $C_c(G)$ and order continuity of the norm

Sketch of proof.

(Inspired by Ben de Pagter and Werner J. Ricker)

1. Every locally compact group contains a σ-compact clopen subgroup $\Longrightarrow$ reduce to the case that G is σ-compact and clopen.
2. For any σ-compact locally compact group G and every countable family $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods of the group identity, there is a compact normal subgroup N of G such that $N \subset \bigcap_{n\in\mathbb{N}} U_n$ and G/N is metrizable $\Longrightarrow$ reduce to the case that G is metrizable.
3. An inductive limit of order continuous Banach lattices is order continuous $\Longrightarrow$ reduce to the case that $E_K := \{f\chi_K \colon f \in E\}$ $(K \subset G)$ compact.
4. Since $C(K)$ is separable when K is metrizable, E_K is separable and hence order continuous.

Generalization of Brainerd and Edwards's result

Theorem

Let E be a translation invariant Banach function space on G and $T : E \to C_c(G)^\sim$ a positive operator that commutes with right translations. If translations are bounded on compact subsets of G and $C_c(G)$ is dense in E, then there exists a positive element $\mu \in C_c(G)^\sim$ such that

$$Tf = \mu * f$$

for all $f \in E$.

Sketch of proof.

1. The restriction of T to $C_c(G)$ is of the form $f \mapsto \mu * f$ for some positive μ in $C_c(G)^\sim$ by Theorem 1.
2. The density of $C_c(G)$ implies that E is order continuous by Theorem 4. So the map $f \mapsto \mu * f$ can be extended to E.

□

Thank you for your attention!