Recent developments in nonlinear Perron-Frobenius theory

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Classical Perron-Frobenius theorem

Theorem (Perron-Frobenius)

If $A \in \mathbb{R}^{n \times n}$ is nonnegative and irreducible, then A has a unique eigenvector (up to scaling) with all positive entries and the corresponding eigenvalue is the spectral radius of A.

A matrix is irreducible if its associated directed graph is strongly connected.

Notation

- $[n] = \{1, \ldots, n\}$.
- $x \geq y$ in $\mathbb{R}^n$ when $x_i \geq y_i$ for all $i \in [n]$.
- The standard cone in $\mathbb{R}^n$ is $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x \geq 0\}$.
- The interior of $\mathbb{R}^n_{\geq 0}$ is $\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i \in [n]\}$.

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A function f is

1. Order-preserving when $x \geq y$ implies that $f(x) \geq f(y)$ for all x, y in the domain.
2. (Multiplicatively) homogeneous if $f(tx) = tf(x)$ for all $t > 0$.
3. Additively homogeneous if $f(x + t\mathbf{1}) = f(x) + t\mathbf{1}$ for all $t \in \mathbb{R}$.

Topical functions

A function $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ that is order-preserving and homogeneous is multiplicatively topical.

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is (additively) topical if T is order-preserving and additively homogeneous.

Any topical $T : \mathbb{R}^n \to \mathbb{R}^n$ corresponds to a multiplicatively topical function

$$f = \exp \circ T \circ \log .$$

Examples of topical functions

Additively topical examples

- Max-plus linear maps
- Min-max-plus operators (e.g., Shapley operators from stochastic game theory)

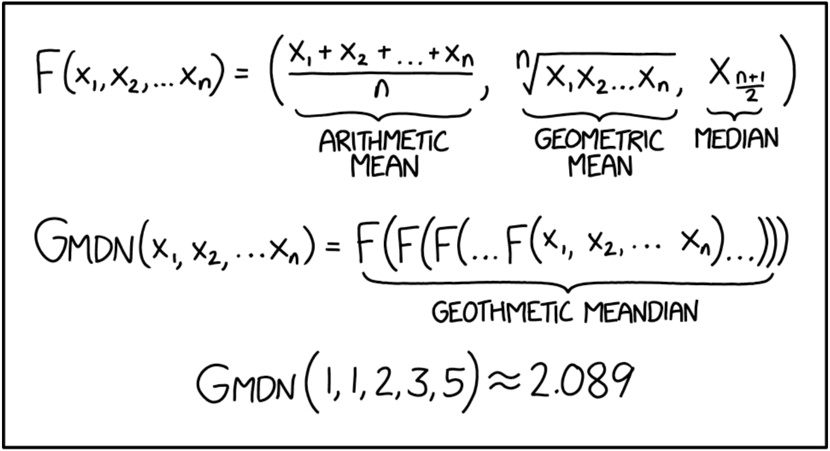
Multiplicatively topical examples

- The homogeneous eigenvalue problem for nonnegative tensors
- Examples from economics and population biology
- The arithmetic-geometric mean function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) \\ \sqrt{x_1 x_2} \end{bmatrix}.$$

The geothmetic meandian

XKCD #2435 by Randall Munroe



STATS TIP: IF YOU AREN'T SURE WHETHER TO USE THE MEAN, MEDIAN, OR GEOMETRIC MEAN, JUST CALCULATE ALL THREE, THEN REPEAT UNTIL IT CONVERGES

Eigenvectors of topical functions

For $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$, the eigenspace of f is

$$E(f) := \{x \in \mathbb{R}^n_{>0} : x \text{ is an eigenvector of } f\}.$$

Note that $E(f)$ only includes eigenvectors with all positive entries.

There might also be eigenvectors on the boundary of the cone $\mathbb{R}^n_{\geq 0}$, but that is not our focus.

Hilbert's projective metric

Hilbert's projective metric on $\mathbb{R}^n_{>0}$ is defined by

$$d_H(x, y) := \log \max_{i,j \in [n]} \left(\frac{y_i \, x_j}{x_i \, y_j} \right).$$

It is a metric on the rays from the origin in $\mathbb{R}^n_{>0}$. Points in the boundary of $\mathbb{R}^n_{\geq 0}$ (i.e., that have zero entries) are infinitely far away.

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If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is order-preserving and homogeneous, then f is nonexpansive with respect to d_H, i.e.,

$$d_H(f(x), f(y)) \leq d_H(x, y) \text{ for all } x, y \in \mathbb{R}^n_{>0}.$$

The hypergraphs $\mathcal{H}_0^-(f)$ and $\mathcal{H}_\infty^+(f)$

For a multiplicatively topical function f, $\mathcal{H}_0^-(f)$ and $\mathcal{H}_\infty^+(f)$ are directed hypergraphs with nodes $[n]$ that were introduced by Akian, Gaubert, and Hochart.

The hyperarcs of $\mathcal{H}_0^-(f)$ are the pairs $(I, \{j\})$ such that $I \subset [n]$, $j \in [n] \setminus I$, and

$$\lim_{t \to \infty} f(\exp(-te_I))_j = 0$$

where exp is the entrywise natural exponential function and $e_I \in \mathbb{R}^n$ has entries

$$(e_I)_i := \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise.} \end{cases}$$

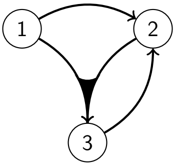
The hyperarcs of $\mathcal{H}_\infty^+(f)$ are $(I, \{j\})$ such that $I \subset [n]$, $j \in [n] \setminus I$ and

$$\lim_{t \to \infty} f(\exp(te_I))_j = \infty.$$

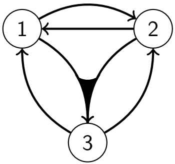
Example

The geothmetic meandian function $F(x) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \sqrt[3]{x_1 x_2 x_3} \\ \text{median}(x_1, x_2, x_3) \end{bmatrix}$ has

$\mathcal{H}_0^-(F)$



$\mathcal{H}_\infty^+(F)$



These show the minimal hyperarcs of $\mathcal{H}_0^-(F)$ and $\mathcal{H}_\infty^+(F)$.

Invariant nodes and reach

A subset $I \subseteq [n]$ is invariant in $\mathcal{H}_0^-(f)$ or $\mathcal{H}_\infty^+(f)$ if there are no hyperarcs $(I, \{j\})$ that originate from I in the hypergraph.

The reach of $J \subset [n]$ in a hypergraph $\mathcal{H}$, denoted $\mathrm{reach}(J, \mathcal{H})$, is the smallest invariant subset of the nodes of $\mathcal{H}$ containing J.

Example

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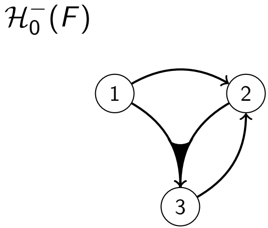

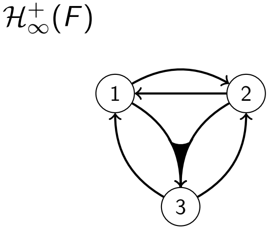

$I = \{2, 3\}$ is invariant in $\mathcal{H}_0^-(F)$, but $\mathcal{H}_\infty^+(F)$ has no invariant subsets.

Super & sub-eigenspaces

For any $\alpha, \beta > 0$, the sub-eigenspace corresponding to α is the set

$$S_\alpha(f) := \{x \in \mathbb{R}^n_{>0} : \alpha x \leq f(x)\}$$

and the super-eigenspace corresponding to β is

$$S^\beta(f) := \{x \in \mathbb{R}^n_{>0} : f(x) \leq \beta x\}.$$

The intersection $S_\alpha^\beta(f) := S_\alpha(f) \cap S^\beta(f)$ is called a slice space.

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Idea: These sets are all invariant under f. If any of these sets is nonempty and bounded in Hilbert's projective metric, then f has a positive eigenvector.

An irreducibility condition

Theorem (Gaubert-Gunawardena, 2004)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. Then all super-eigenspaces $S^\beta(f)$ are bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if $\text{reach}(J, \mathcal{H}^+_\infty(f)) = [n]$ *for every nonempty $J \subsetneq [n]$.*

A corresponding condition involving the hypergraph $\mathcal{H}^-_0(f)$ is equivalent to all sub-eigenspaces of f being d_H-bounded.

Bounded slice spaces

Theorem (Akian-Gaubert-Hochart, 2020)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. All slice spaces $S_\alpha^\beta(f)$ are bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if

$$\operatorname{reach}(J, \mathcal{H}^+_\infty(f)) = [n] \text{ or } \operatorname{reach}(J^c, \mathcal{H}^-_0(f)) = [n]$$

for every nonempty $J \subsetneq [n]$.

Nonempty & bounded eigenspace

Theorem (Lemmens-L-Nussbaum, 2018)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. The eigenspace $E(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if for every nonempty $J \subsetneq [n]$, there exists $x \in \mathbb{R}^n$ such that

$$\max_{j \in J} \frac{f(x)_j}{x_j} < \min_{i \in J^c} \frac{f(x)_i}{x_i}.$$

In general the conditions of these three theorems are progressively more difficult to check.

Upper & lower Collatz-Wielandt numbers

The upper Collatz-Wielandt number for f is

$$r(f) := \inf\{\beta > 0 : S^\beta(f) \text{ is nonempty}\},$$

and the lower Collatz-Wielandt number for f is

$$\lambda(f) := \sup\{\alpha > 0 : S_\alpha(f) \text{ is nonempty}\}.$$

Alternatively, $r(f)$ is the infimum of the super-eigenvalues and $\lambda(f)$ is the supremum of the sub-eigenvalues.

If $E(f)$ is nonempty, then $\lambda(f) = r(f)$, but the converse is not always true.

The upper Collatz-Wielandt number $r(f)$ is equal to the *cone spectral radius*, i.e., the largest eigenvalue of f as a map on $\mathbb{R}^n_{\geq 0}$.

Boundary projections

For $\alpha \in [0, \infty]$ and $J \subseteq [n]$, let P^J_α be the projection

$$P^J_\alpha(x)_j := \begin{cases} x_j & \text{if } j \in J \\ \alpha & \text{otherwise.} \end{cases}$$

For any order-preserving homogeneous function $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$, we define

$$f^J_0 := P^J_0 f P^J_0 \qquad \text{and} \qquad f^J_\infty := P^J_\infty f P^J_\infty.$$

Both $f^J_0 : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ and $f^J_\infty : (0, \infty]^n \to (0, \infty]^n$ are order-preserving and homogeneous functions.

Bounded nonempty eigenspaces - revisited

Theorem (L, 2023)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. The eigenspace $E(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if

$$r(f_0^J) < \lambda(f_\infty^{[n]\setminus J})$$

for every nonempty $J \subsetneq [n]$.

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Theorem (L, 2023)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. The eigenspace $E(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if

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Lemma

For $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous,

- $\text{reach}(J^c, \mathcal{H}_0^-(f)) = [n] \iff r(f_0^J) = 0$.
- $\text{reach}(J, \mathcal{H}_\infty^+(f)) = [n] \iff \lambda(f_\infty^{[n]\setminus J}) = \infty$.

So you can check the hypergraphs first, and only check the Collatz-Wielandt numbers for J where the reach condition fails.

Example

For geothmetic meandian function $F(x) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \sqrt[3]{x_1 x_2 x_3} \\ \text{median}(x_1, x_2, x_3) \end{bmatrix}$:

- $F_\infty^{\{2,3\}}(x) = P_\infty^{\{2,3\}} F P_\infty^{\{2,3\}}(x) = \begin{bmatrix} \infty \\ \infty \\ \max(x_2, x_3) \end{bmatrix}$,

$$\lambda(F_\infty^{\{2,3\}}) = \infty.$$

- $F_0^{\{1,3\}}(x) = P_0^{\{1,3\}} F P_0^{\{1,3\}}(x) = \begin{bmatrix} \frac{1}{3}(x_1 + x_3) \\ 0 \\ \min(x_1, x_3) \end{bmatrix}$,

$$r(F_0^{\{1,3\}}) = \frac{1}{6}(1 + \sqrt{13})$$

Convex maps

1. Checking that $E(f)$ is nonempty and bounded requires checking an exponential number of subsets $J \subsetneq [n]$. This can be reduced dramatically if the additively topical map $\log \circ f \circ \exp$ is convex.

2. In addition, if $\log \circ f \circ \exp$ is convex and real analytic, or convex and piecewise affine, then we can give complete necessary and sufficient conditions for $E(f)$ to be nonempty.

Unique fixed points of real analytic nonexpansive maps

Theorem (L, 2023)

Let X be a real Banach space with the fixed point property. Let $f : X \to X$ be nonexpansive and real analytic. If f has more than one fixed point, then the set of fixed points of f is unbounded.

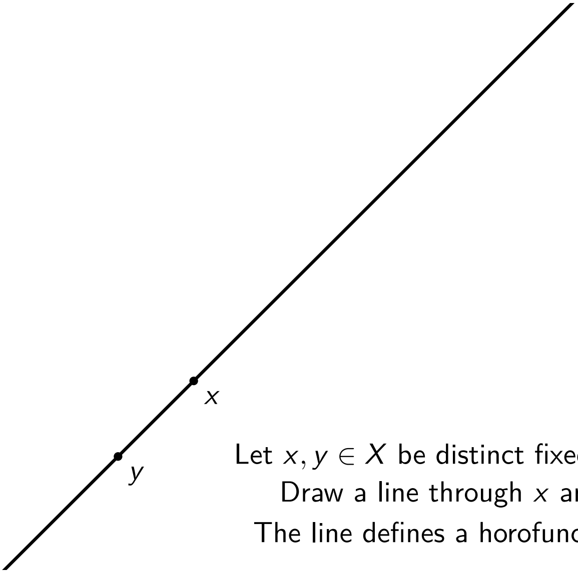
Corollary

If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is order-preserving, homogeneous, and real analytic, then f has a unique eigenvector (up to scaling) if and only if

$$r(f_0^J) < \lambda(f_\infty^{[n]\setminus J})$$

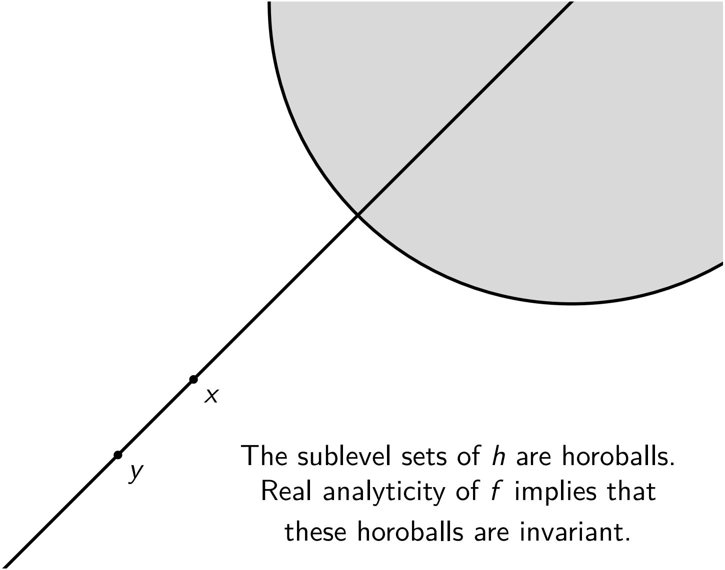
for every nonempty $J \subsetneq [n]$.

Intuition for uniqueness



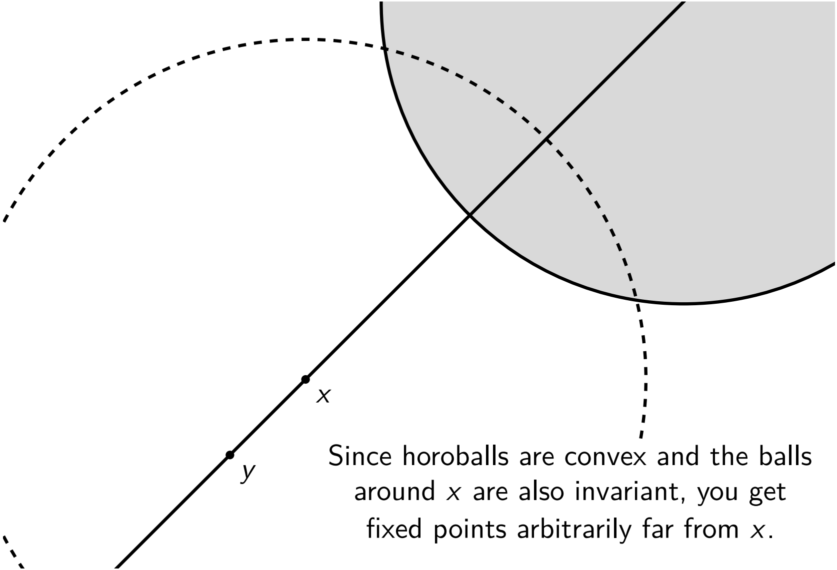
Let $x, y \in X$ be distinct fixed points.
Draw a line through x and y.
The line defines a horofunction h.

Intuition for uniqueness



The sublevel sets of h are horoballs.
Real analyticity of f implies that
these horoballs are invariant.

Intuition for uniqueness



Since horoballs are convex and the balls around x are also invariant, you get fixed points arbitrarily far from x.

Thanks & references

Thanks for your attention!

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