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The essential spectrum, norm, and spectral radius of abstract multiplication operators

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Motivation

Main reason of the paper of the talk: Recently the essential norm $\|T\|_e$ in the Calkin algebra has been considered for certain concrete Banach sequence spaces and L_p-spaces. See (Voight, 2022) and (Castillo, 2022) for more references. The novelty of these latter papers was that they combined the results for non-atomic measure spaces and atomic ones. These papers were not aware of the earlier more general results results by me (for the non-atomic case) and Arend-Sourour (for more results about the essential spectrum). In this talk we will extend the recent result to any Banach lattice.

Definitions and notations

E will denote a Banach lattice and the collection of band preserving operators is equal to the center $Z(E)$, where $Z(E) = \{T \in \mathcal{L}(E) : |T| \leq \lambda I \text{ for some } \lambda\}$ and $\|T\| = \inf\{\lambda : |T| \leq \lambda I\}$ for $T \in Z(E)$. We proved in 1980 that $Z(E)$ is a full sub-algebra of the algebra $\mathcal{L}(E)$ of norm bounded operators, i.e., $T \in Z(E)$ is invertible in $\mathcal{L}(E)$ if and only if there exists $c > 0$ such that $|T| \geq cI$. In this talk the essential spectrum $\sigma_e(T)$ denotes the spectrum of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$. We proved in 1980 that $\sigma_e(T) = \sigma(T)$ for $T \in Z(E)$ with E a non-atomic Banach lattice. Arendt and Sourour gave a new proof in 1986 and characterized Fredholm properties of operators in $Z(E)$.

Let E be a complex Banach lattice. Recall that $a \in E$ is called an atom, if $|x| \leq |a|$ implies that $x = \lambda a$ for some scalar λ, i.e., $\{a\}^{dd} = \{\lambda a : \lambda \in \mathbb{C}\}$. The following theorem was proved by the author in 1980.

Theorem

Let E be a complex non-atomic Banach lattice and $T \in Z(E)$. Then $\sigma(T) = \sigma_e(T)$ and thus $r_e(T) = r(T) = \|T\|$.

The proof of this theorem, as well as the following theorem, depend on Lemma 1.10 of that paper, which is incorrect as stated. Fortunately the proof is correct if we weaken the statement slightly in the preceding lemma. The corrected Lemma 1.10 of says that if E is a non-atomic complex Banach lattice, then for all $w \neq 0$ there exist $w_n \in \{w\}^{dd}$ with $\|w_n\| = 1$ such that $w_n \to 0$ in the weak topology $\sigma(E, E^*)$.

Using this corrected statement of the lemma, we can similarly prove

Theorem

Let E be a complex non-atomic Banach lattice and $T \in Z(E)$. Then $\|T\|_e = \|T\| = r_e(T)$.

Proof.

Let $0 < c < \|T\|$. Then $(|T| - cI)^+ > 0$. Therefore there exists $0 < u \in E$ such that $v = (|T| - cI)^+ u > 0$. Now $(|T| - cI)v = ((|T| - cI)^+)^2 u \geq 0$. Therefore $|T|v \geq cv$. By order continuity of T this implies that $|Tw| \geq c|w|$ for all $w \in \{v\}^{dd}$. In particular $\|Tw\| \geq c\|w\|$ for all $w \in \{v\}^{dd}$. Now by the corrected (as indicated above) Lemma 1.10 of the 1980 paper there exist $w_n \in \{v\}^{dd}$ with $\|w_n\| = 1$ such that $w_n \to 0$ in the weak topology $\sigma(E, E^*)$. Let now $K \in \mathcal{K}(E)$. Then, by compactness of K, we have that $\|Kw_n\| \to 0$ as $n \to \infty$. From this it follows that

$$\begin{aligned}\|T - K\| &\geq \limsup \|Tw_n - Kw_n\| \\ &\geq \limsup |\|Tw_n\| - \|Kw_n\|| = \limsup \|Tw_n\| \geq c.\end{aligned}$$

As this holds for all $0 < c < \|T\|$, it follows that $\|T - K\| \geq \|T\|$. Hence $\|T\|_e \geq \|T\|$ and thus $\|T\|_e = \|T\|$. □

To deal with uncountably many cluster points we introduce the following. Let A be a non-empty set and $f : A \to \mathbb{C}$ a function. Denote by $\mathcal{F}$ the Fréchet filter on A, i.e., a subset B of A is in $\mathcal{F}$ if B^c is a finite set.

Definition

A point $z \in \mathbb{C}$ is called a Fréchet cluster point of f if for all $\epsilon > 0$ and all $B \in \mathcal{F}$ there is an element $a \in B$ such that $|f(a) - z| < \epsilon$.

In this case we will also say that z is an $\mathcal{F}$-cluster point of f on A. It is also known that the set of all $\mathcal{F}$-cluster points of f is a closed subset of $\mathbb{C}$. Therefore we have the following proposition.

Proposition

Let A be a non-empty set and $f : A \to \mathbb{C}$ a bounded function. Then there exists an $\mathcal{F}$-cluster point of f with largest and smallest modulus.

We will now describe the modulus this largest and smallest $\mathcal{F}$-cluster point.

Definition

Let A be a non-empty set and $f : A \to \mathbb{R}$ a bounded function. Then the $\mathcal{F}$-limit superior of f is defined as

$$\limsup_{\mathcal{F}} f = \lim_{F \in \mathcal{F}} \sup f[F] = \inf_{F \in \mathcal{F}} \sup f[F]$$

and the $\mathcal{F}$-limit inferior of f is defined similarly.

Proposition

Let A be a non-empty set and $f : A \to \mathbb{C}$ a bounded function with λ an $\mathcal{F}$-cluster point of f of largest modulus. Then

$$|\lambda| = \limsup_{\mathcal{F}} |f|.$$

Let E be an infinite dimensional (complex) Banach lattice throughout this section. Denote by A the set of all positive atoms in E of norm one. Then E is called an atomic Banach lattice if the band A^{dd} generated by A equals E. By P_a we denote the band projection from E onto $\{a\}^{dd}$. Every $T \in Z(E)$ can be represented as a multiplication operator, where for each $a \in A$ there exists $\lambda_a \in \mathbb{C}$ such that $Ta = \lambda_a a$. The following lemma is well-known in the countable case.

Lemma

Let E be an atomic Banach lattice with, as above, the set A the set of all positive atoms of norm one. Let $T \in Z(E)$ as above. Then $T \in \mathcal{K}(E)$ if and only if

$$\lim_{\mathcal{F}} |\lambda_a| = \limsup_{\mathcal{F}} |\lambda_a| = 0.$$

Theorem

Let E be an atomic Banach lattice with, as above, the set A the set of all positive atoms of norm one. Let $T \in Z(E)$. Then the essential spectrum $\sigma_e(T)$ is given by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is a Fréchet cluster point of the function } a \mapsto \lambda_a\}.$$

Moreover, the essential spectral radius $r_e(T)$ of T is given by

$$r_e(T) = \limsup_{\mathcal{F}} |\lambda_a|,$$

and the essential norm $\|T\|_e$ of T satisfies $\|T\|_e = r_e(T)$.

Theorem

Let E be a (complex) Banach lattice and $T \in Z(E)$. Then

$$\sigma_e(T) = \sigma_e(T_A) \cup \sigma(T_{A^d})$$

and thus

$$r_e(T) = \max\{r_e(T_A), \|T_{A^d}\|\}.$$

Moreover $\|T\|_e = r_e(T)$.

Here T_A denotes the atomic part of T and T_{A^d} denotes the non-atomic part of T.

In 1969 Gustafson observed that if $\sigma_e(A+B) = \sigma_e(A)$ for all bounded self-adjoint operators A on a Hilbert space, for a given self-adjoint B, then B is compact. The following theorem is an order analogue of this.

Theorem

Let E be a (complex) Banach lattice and $T \in Z(E)$ such that $\sigma_e(I+T) = \sigma_e(I)$. Then T is compact.

We could have phrased the above result alternatively as: If $T \in Z(E)$ is essentially quasi-nilpotent, then T is compact.

To describe the essential norm of a disjointness preserving operator the Hausdorff (or ball) measure of non-compactness is an essential tool. Recall that for a norm bounded subset D of E the Hausdorff measure of non- compactness is defined by

$$\beta(D) = \inf\{\delta > 0 : \exists f_1, \cdots, f_n \in E \text{ such that } D \subset \bigcup_{i=1}^{n} B(f_i, \delta)\},$$

where $B(f, \delta)$ denotes the open ball with center f and radius δ. For a bounded operator $T : E \to E$ we then the Hausdorff measure $\beta(T)$ by $\beta(T(B_E))$.

Theorem

(Schep 1989) Let E be a complex Banach lattice such that E^* is non-atomic and let $T : E \to E$ be a normbounded disjointness preserving operator. Then $\|T\|_e = \beta(T) = \|T\|$.

It is easy to see that we can not replace the condition that E^* is non-atomic by the weaker condition that E is non-atomic.

For the atomic case we have the following slightly more general result than a result of Goldenstein and Markus.

Theorem

Let E be an atomic Banach with order continuous norm and let $T : E \to E$ be a norm bounded operator. Then $\beta(T) = \|T\|_e$.

Recall that the measure of non-semicompactness $\rho(D)$ of D is defined as

$$\rho(D) = \inf\{\delta > 0 : \exists 0 \leq u \in E \text{ such that } D \subset [-u, u]_{\mathbb{C}} + \delta B_E\}.$$

Proposition

Let E be an atomic Banach with order continuous norm and let $D \subset E$. Then $\rho(D) = \beta(D)$.

We now provide a formula for $\rho(D)$, which allows us to make explicit calculations in concrete examples.

Theorem

Let E be an atomic Banach with order continuous norm and let $D \subset E$ a norm bounded subset. Then

$$\begin{aligned}\rho(D) &= \sup\{\lim_{n\to\infty}(\sup(\|P_n x\| : x \in D)) : P_n \downarrow 0, P_n \text{ a band projection}\}\\ &= \inf\{\sup(\|P_B x\| : x \in D) : B \subset A, B^c \text{ finite}\}\\ &= \lim_{B\in\mathcal{G}}(\sup((\|P_B x\| : x \in D)),\end{aligned}$$

where $\mathcal{G}$ is the Fréchet filter on A, where A is as before the set of atoms.

For a norm bounded operator $T : E \to E$ we now define $\rho(T) = \rho(T(B_E))$. Applying the above proposition and theorems to $D = T(B_E)$ we get

Theorem

Let E be an atomic Banach with order continuous norm and let $T : E \to E$ be norm bounded operator. Then

$$\|T\|_e = \beta(T) = \rho(T) = \sup\{\lim_{n\to\infty} \|P_n T\| : P_n \downarrow 0, P_n \text{ a band projection}\}$$
$$= \lim_{B\in\mathcal{G}} \|P_B T\|.$$

In case A is countable with $A = \{a_n : n \geq 1\}$, we have

$$\|T\|_e = \beta(T) = \rho(T) = \lim_{n\to\infty} \|P_n T\|,$$

where P_n is the band projection on $\{a_k : k \geq n\}^{dd}$.

To be able to try to combine the atomic and non-atomic case we have to make additional assumptions. The most natural one is that E has order continuous norm. In this case we have a decomposition $E = E_A \oplus E_{A^d}$ into an atomic part and a non-atomic part and the dual of the non-atomic is then also non-atomic. The atomic part T_A is then defined as $P_A T P_A$ and the non-atomic part as $P_{A^d} T P_{A^d}$ of T. In the case of the center this would give a block diagonal decomposition of T, but here we would expect a two by two block matrix decomposition of T, but under the present assumption we actually have a block triangular decomposition.

Proposition

Let E and F be Banach lattices with order continuous norm, E non-atomic and F atomic. If $T : E \to F$ is disjointness preserving, then $T = 0$.

The previous observation reduces the problem to lower triangular block matrix disjointness preserving operators. Easy examples show that the essential spectrum and norm of T will depend on the off-diagonal block, so no simple expressions like in the center case will be available here. Assuming that the off-diagonal block is zero and that $E = E_A \oplus_p E_{A^d}$ one can describe the essential spectrum of T in terms of the atomic part and non-atomic part. For the essential norm right now we need to make even more assumptions and the results are not final.