Orthogonality in ordered vector spaces

Anil Kumar Karn
National Institute of Science Education and Research
Bhubaneswar, India

Lattice structure in C*-algebras?

Theorem. (Kakutani- Gelfand-Naimark, 1941/ 1943) The self-adjoint part of a unital commutative C*-algebra A is precisely an AM-space upto a unital order isomorphism.

Lattice structure in C*-algebras?

Theorem. (Kakutani- Gelfand-Naimark, 1941/ 1943) The self-adjoint part of a unital commutative C*-algebra A is precisely an AM-space upto a unital order isomorphism.

Theorem. (Sherman, 1951). Let A be a C*-algebra. Then A_{sa} is a vector lattice if and only if A is commutative.

Lattice structure in C*-algebras?

Theorem. (Kakutani- Gelfand-Naimark, 1941/ 1943) The self-adjoint part of a unital commutative C*-algebra A is precisely an AM-space upto a unital order isomorphism.

Theorem. (Sherman, 1951). Let A be a C*-algebra. Then A_{sa} is a vector lattice if and only if A is commutative.

Kadison's Anti-lattice Theorem (1951). Let H be a complex Hilbert space and consider the real ordered vector space $B(H)_{sa}$. For $S, T \in B(H)_{sa}$, we have $S \wedge T$ exists in $B(H)_{sa}$ if and only if S and T are comparable.

Lattice structure in C*-algebras?

Theorem. (Kakutani- Gelfand-Naimark, 1941/ 1943) The self-adjoint part of a unital commutative C*-algebra A is precisely an AM-space upto a unital order isomorphism.

Theorem. (Sherman, 1951). Let A be a C*-algebra. Then A_{sa} is a vector lattice if and only if A is commutative.

Kadison's Anti-lattice Theorem (1951). Let H be a complex Hilbert space and consider the real ordered vector space $B(H)_{sa}$. For $S, T \in B(H)_{sa}$, we have $S \wedge T$ exists in $B(H)_{sa}$ if and only if S and T are comparable.

Shouldn't we expect any lattice-like structure in non-commutative C*-algebras?

Orthogonality in C^*-algebras

Let A be a C^*-algebra. We shall say that $a, b \in A$ are algebraically orthogonal ($a \perp^a b$), if $a^* b = 0 = ab^*$. In particular, when a and b are self-adjoint, then $a \perp^a b$, if and only if $ab = 0$.

Orthogonality in C^*-algebras

Let A be a C^*-algebra. We shall say that $a, b \in A$ are algebraically orthogonal ($a \perp^a b$), if $a^* b = 0 = ab^*$. In particular, when a and b are self-adjoint, then $a \perp^a b$, if and only if $ab = 0$.

Presence of algebraic and norm orthogonality in C^*-algebras. Let A be a C^*-algebra. Then for each $a \in A_{sa}$, there exists a unique pair $a^+, a^- \in A^+$ such that

- $a = a^+ - a^-$; and
- $a^+ a^- = 0$.

Orthogonality in C^*-algebras

Let A be a C^*-algebra. We shall say that $a, b \in A$ are algebraically orthogonal ($a \perp^a b$), if $a^* b = 0 = ab^*$. In particular, when a and b are self-adjoint, then $a \perp^a b$, if and only if $ab = 0$.

Presence of algebraic and norm orthogonality in C^*-algebras. Let A be a C^*-algebra. Then for each $a \in A_{sa}$, there exists a unique pair $a^+, a^- \in A^+$ such that

- $a = a^+ - a^-$; and
- $a^+ a^- = 0$.

  For $x \in A$, we define $|x| := (x^* x)^{\frac{1}{2}}$. In particular, for $x \in A_{sa}$, $|x| = (x^2)^{\frac{1}{2}}$ and for $x \in A^+$, $|x| = x$.
- $|a| = a^+ + a^-$; and
- $\|a\| = \max\{\|a^+\|, \|a^-\|\}$.

Orthogonality in C^*-algebras

Properties of algebraic orthogonality in a C*-algebra:
Let A be a C*-algebra and let $a, b, c \in A_{sa}$. Then

1. $a \perp^a 0$;
2. $a \perp^a b$ implies $b \perp^a a$;
3. $a \perp^a b$ and $a \perp^a c$ imply $a \perp^a (kb + c)$ for all $k \in \mathbb{R}$;
4. If $a \perp^a b$ and if $|c| \leq |b|$, then $a \perp^a c$;
5. For each $a \in A_{sa}$, there exist unique $a^+, a^- \in A^+$ with $a^+ \perp^a a^-$ such that $a = a^+ - a^-$.
   (We also have $|a| = a^+ + a^-$.)

Orthogonality in vector lattices

Let (L, L^+) be a vector lattice. We write $u \wedge v$ for $\inf\{u, v\}$ and $u \vee v$ for $\sup\{u, v\}$. We define $u^+ := u \vee 0$, $u^- := (-u) \vee 0$ and $|u| := u \vee (-u)$ for all $u \in L$. Then $u = u^+ - u^-$ and $|u| = u^+ + u^-$. For $u, v \in L$, we say that u is orthogonal to v if $|u| \wedge |v| = 0$. In this case, we write $u \perp^\ell v$.

Recall that $|u + v| \leq |u| + |v|$ and that

$$u \wedge v = \frac{1}{2}\{u + v - |u - v|\}$$

and

$$u \vee v = \frac{1}{2}\{u + v + |u - v|\}$$

for all $u, v \in L$.

Orthogonality in vector lattices

Properties of orthogonality in a vector lattice:

Let L be a vector lattice and let $u, v, w \in L$. Then

1. $u \perp^{\ell} 0$;
2. $u \perp^{\ell} v$ implies $v \perp^{\ell} u$;
3. $u \perp^{\ell} v$ and $u \perp^{\ell} w$ imply $u \perp^{\ell} (kv + w)$ for all $k \in \mathbb{R}$;
4. If $u \perp^{\ell} v$ and if $|w| \leq |v|$, then $u \perp^{\ell} w$;
5. For each $u \in L$, there exist unique $u^+, u^- \in L^+$ with $u^+ \perp^{\ell} u^-$ such that $u = u^+ - u^-$.
   (We also have $|u| = u^+ + u^-$.)

Orthogonality in ordered vector spaces

Definition. Let (V, V^+) be a real ordered vector space. Assume that $\perp$ is a binary relation in V such that for $u, v, w \in V$, we have

1. $u \perp 0$;
2. $u \perp v$ implies $v \perp u$;
3. $u \perp v$ and $u \perp w$ imply $u \perp (kv + w)$ for all $k \in \mathbb{R}$;
4. For each $u \in L$, there exist unique $u^+, u^- \in L^+$ with $u^+ \perp u^-$ such that $u = u^+ - u^-$.
   Let us put $u^+ + u^- := |u|$.
5. If $u \perp v$ and if $|w| \leq |v|$, then $u \perp w$.

Then V is called an absolutely ordered vector space.

Orthogonality in ordered vector spaces

Proposition. Let V be an absolutely ordered space and let $u, v \in V$.

1. $|u - v| = u + v$ if and only if $u, v \in V^+$ with $u \perp v$.
2. $u \perp v$ if and only if $|u \pm v| = |u| + |v|$.

Orthogonality in ordered vector spaces

Proposition. Let V be an absolutely ordered space and let $u, v \in V$.

1. $|u - v| = u + v$ if and only if $u, v \in V^+$ with $u \perp v$.
2. $u \perp v$ if and only if $|u \pm v| = |u| + |v|$.

Theorem. Let V be an absolutely ordered space and let $u, v, w \in V$.

1. $|u| = u$ if and only if $u \in V^+$.
2. $|u| \pm u \in V^+$.
3. $|ku| = |k||u|$ for all $k \in \mathbb{R}$.
4. If $|u - v| = u + v$ and if $0 \leq w \leq v$, then $|u - w| = u + w$.
5. If $|u - v| = u + v$ and $|u - w| = u + w$, then $|u - |v \pm w|| = u + |v \pm w|$.

A substitute

Theorem. Let V be an absolutely ordered space. Then the following statements are equivalent:

1. $|u+v| \leq |u| + |v|$ for all $u, v \in V$;
2. V is a vector lattice.

A substitute

Theorem. Let V be an absolutely ordered space. Then the following statements are equivalent:

1. $|u+v| \leq |u| + |v|$ for all $u, v \in V$;
2. V is a vector lattice.

Definition. Let V be an absolutely ordered vector space. Then $w \in V$ is said to be an ortho-infimum of $u, v \in V$ if

1. $w \leq u$ and $w \leq v$; and
2. $(u - w) \perp (v - w)$.

A substitute

Theorem. Let V be an absolutely ordered space. Then the following statements are equivalent:

1. $|u+v| \leq |u| + |v|$ for all $u, v \in V$;
2. V is a vector lattice.

Definition. Let V be an absolutely ordered vector space. Then $w \in V$ is said to be an ortho-infimum of $u, v \in V$ if

1. $w \leq u$ and $w \leq v$; and
2. $(u - w) \perp (v - w)$.

Similarly, $x \in V$ is said to be an ortho-supremum of u and v, if

1. $u \leq x$ and $v \leq x$; and
2. $(x - u) \perp (x - v)$.

Orthogonality in ordered vector spaces

Theorem. Let V be an absolutely ordered vector space. For $u, v \in V$ we set

$$u \dot{\wedge} v := \frac{1}{2}\{u + v - |u - v|\}$$

and

$$u \dot{\vee} v := \frac{1}{2}\{u + v + |u - v|\}.$$

Then the ortho-infimum of $u, v \in V$ is uniquely determined as $u \dot{\wedge} v$ and the ortho-supremum of $u, v \in V$ is uniquely determined as $u \dot{\vee} v$.

Orthogonality in ordered vector spaces

Theorem. Let V be an absolutely ordered vector space. For $u, v \in V$ we set

$$u \dot{\wedge} v := \frac{1}{2}\{u + v - |u - v|\}$$

and

$$u \dot{\vee} v := \frac{1}{2}\{u + v + |u - v|\}.$$

Then the ortho-infimum of $u, v \in V$ is uniquely determined as $u \dot{\wedge} v$ and the ortho-supremum of $u, v \in V$ is uniquely determined as $u \dot{\vee} v$.

Theorem. In a vector lattice, the ortho-infimum is the infimum and the ortho-supremum is the supremum.

Orthogonality and norm

Orthogonality and norm

Definition. Let V be a real normed linear space and let $u, v \in V$. For $1 \leq p \leq \infty$, we say that u is p-orthogonal to v, ($u \perp_p v$), if for all $k \in \mathbb{R}$, we have

$$\|u + kv\| = \begin{cases} \left(\|u\|^p + \|kv\|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max\left(\|u\|, \|kv\|\right), & \text{if } p = \infty. \end{cases}$$

Orthogonality and norm

Definition. Let V be a real normed linear space and let $u, v \in V$. For $1 \leq p \leq \infty$, we say that u is p-orthogonal to v, ($u \perp_p v$), if for all $k \in \mathbb{R}$, we have

$$\|u + kv\| = \begin{cases} \left(\|u\|^p + \|kv\|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max\left(\|u\|, \|kv\|\right), & \text{if } p = \infty. \end{cases}$$

Definition Let $1 \leq p \leq \infty$ and $S \subset V$. We say that $\perp_p$ is additive in S, if for $u, v, w \in S$ with $u \perp_p v$ and $u \perp_p w$, we have $u \perp_p (v + w)$.

Orthogonality and norm

Definition. Let V be a real normed linear space and let $u, v \in V$. For $1 \leq p \leq \infty$, we say that u is p-orthogonal to v, ($u \perp_p v$), if for all $k \in \mathbb{R}$, we have

$$\|u + kv\| = \begin{cases} \left(\|u\|^p + \|kv\|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max\left(\|u\|, \|kv\|\right), & \text{if } p = \infty. \end{cases}$$

Definition Let $1 \leq p \leq \infty$ and $S \subset V$. We say that $\perp_p$ is additive in S, if for $u, v, w \in S$ with $u \perp_p v$ and $u \perp_p w$, we have $u \perp_p (v + w)$.
We say that S is a p-orthogonal set in V, if $u \perp_p v$ whenever $u, v \in S$ and $u \neq v$.

Orthogonality and norm

Definition. Let V be a real normed linear space and let $u, v \in V$. For $1 \leq p \leq \infty$, we say that u is p-orthogonal to v, ($u \perp_p v$), if for all $k \in \mathbb{R}$, we have

$$\|u + kv\| = \begin{cases} \left(\|u\|^p + \|kv\|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max\left(\|u\|, \|kv\|\right), & \text{if } p = \infty. \end{cases}$$

Definition Let $1 \leq p \leq \infty$ and $S \subset V$. We say that $\perp_p$ is additive in S, if for $u, v, w \in S$ with $u \perp_p v$ and $u \perp_p w$, we have $u \perp_p (v + w)$.
We say that S is a p-orthogonal set in V, if $u \perp_p v$ whenever $u, v \in S$ and $u \neq v$.
We say that S is total in V, if the linear span of S is dense in V.

Geometric orthogonality in ℓ_p-spaces

Theorem. Let $1 \leq p < \infty$. A Banach space V is isometrically isomorphic to $\ell_p(I)$, if and only if $\perp_p$ is additive in V and there exists a subset U of V of cardinality I such that U is p-orthonormal and is total in V. For $p = \infty$, we need to replace $\ell_\infty(I)$ by $c_0(I)$.

Geometric orthogonality in ℓ_p-spaces

Definition Let $(V, V^+, |\cdot|)$ be a real ordered space and let $\|\cdot\|$ be a norm on V. Then $(V, V^+, \|\cdot\|)$ is said to be an order smooth p-normed space, for $1 \leq p \leq \infty$, if it satisfies the following conditions:

$(O.p.1)$: For $u \leq v \leq w$ in U, we have

$$\|v\| \leq \begin{cases} (\|u\|^p + \|w\|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max(\|u\|, \|w\|) & \text{if } p = \infty; \end{cases}$$

$(O.p.2)$: if $v \in V$ and $\epsilon > 0$, then there exist $v_1, v_2 \in V^+$ with $v = v_1 - v_2$ such that

$$\|v\| + \epsilon \geq \begin{cases} (\|v_1\|^p + \|v_2\|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max(\|v_1\|, \|V_2\|) & \text{if } p = \infty; \end{cases}$$

Note that an order unit space is an order smooth ∞-normed space.

Geometric orthogonality in ℓ_p-spaces

Theorem. Let $1 \leq p < \infty$ and assume that $(V, V^+, \|\cdot\|)$ be a norm complete order smooth p-normed space. If $\perp_p$ is additive in V^+ and if U is a total p-orthonormal set in V^+, then V is isometrically order isomorphic to $\ell_p(I)$ where I is the cardinality of U. For $p = \infty$ we need to replace $\ell_\infty(I)$ by $c_0(I)$.

Orthogonality in C*-algebras

Example Consider the 3-dimensional real sequence space $V = \ell_\infty^3$. We have $u = \langle 1, \frac{1}{2}, 0 \rangle$ and $v = \langle 0, \frac{1}{3}, 1 \rangle$ in V such that $u \perp_\infty v$. However, $uv \neq 0$ in the coordinate-wise multiplication.

Orthogonality in C*-algebras

Example Consider the 3-dimensional real sequence space $V = \ell_\infty^3$. We have $u = \langle 1, \frac{1}{2}, 0\rangle$ and $v = \langle 0, \frac{1}{3}, 1\rangle$ in V such that $u \perp_\infty v$. However, $uv \neq 0$ in the coordinate-wise multiplication.

Definition. Let A be a C*-algebra. For $a, b \in A^+$, we say that a is absolutely ∞-orthogonal to b ($a \perp_\infty^a b$), if $[0, a] \perp_\infty [0, b]$.

Orthogonality in C*-algebras

Example Consider the 3-dimensional real sequence space $V = \ell_\infty^3$. We have $u = \langle 1, \frac{1}{2}, 0 \rangle$ and $v = \langle 0, \frac{1}{3}, 1 \rangle$ in V such that $u \perp_\infty v$. However, $uv \neq 0$ in the coordinate-wise multiplication.

Definition. Let A be a C*-algebra. For $a, b \in A^+$, we say that a is absolutely ∞-orthogonal to b ($a \perp_\infty^a b$), if $[0, a] \perp_\infty [0, b]$. More generally, if $(V, V^+, \| \cdot \|)$ is an order smooth p-normed space for $1 \leq p \leq \infty$, then for $u, v \in V^+$, we say that u is absolutely p-orthogonal to v ($u \perp_p^a v$), if $[0, u] \perp_p [0, v]$.

Orthogonality in C*-algebras

Example Consider the 3-dimensional real sequence space $V = \ell_\infty^3$. We have $u = \langle 1, \frac{1}{2}, 0 \rangle$ and $v = \langle 0, \frac{1}{3}, 1 \rangle$ in V such that $u \perp_\infty v$. However, $uv \neq 0$ in the coordinate-wise multiplication.

Definition. Let A be a C*-algebra. For $a, b \in A^+$, we say that a is absolutely ∞-orthogonal to b ($a \perp_\infty^a b$), if $[0, a] \perp_\infty [0, b]$. More generally, if $(V, V^+, \|\cdot\|)$ is an order smooth p-normed space for $1 \leq p \leq \infty$, then for $u, v \in V^+$, we say that u is absolutely p-orthogonal to v ($u \perp_p^a v$), if $[0, u] \perp_p [0, v]$.

Theorem. Let a and b be any two positive elements of a C^*-algebra A. Then $a \perp^a b$ if and only if $a \perp_\infty^a b$.

Geometric orthogonality in ordered vector spaces

Let $(V, V^+, |\cdot|)$ be an absolutely ordered space and let $\|\cdot\|$ be a norm on V. Then $(V, V^+, |\cdot|, \|\cdot\|)$ is said to be an absolute order smooth p-normed space, for $1 \le p \le \infty$, if it satisfies the following conditions:

$(O.p.1)$: For $u \le v \le w$ in V, we have

$$\|v\| \le \begin{cases} (\|u\|^p + \|w\|^p)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \max(\|u\|, \|w\|) & \text{if } p = \infty; \end{cases}$$

$(O.\perp_p.1)$: if $u, v \in V^+$ with $u \perp v$, then $u \perp_p^a v$; and

$(O.\perp_p.2)$: if $u, v \in V^+$ with $u \perp_p^a v$, then $u \perp v$.

Geometric orthogonality in ordered vector spaces

Let $(V, V^+, |\cdot|)$ be an absolutely ordered space and let $\|\cdot\|$ be a norm on V. Then $(V, V^+, |\cdot|, \|\cdot\|)$ is said to be an absolute order smooth p-normed space, for $1 \leq p \leq \infty$, if it satisfies the following conditions:

$(O.p.1)$: For $u \leq v \leq w$ in V, we have

$$\|v\| \leq \begin{cases} (\|u\|^p + \|w\|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max(\|u\|, \|w\|) & \text{if } p = \infty; \end{cases}$$

$(O. \perp_p .1)$: if $u, v \in V^+$ with $u \perp v$, then $u \perp_p^a v$; and

$(O. \perp_p .2)$: if $u, v \in V^+$ with $u \perp_p^a v$, then $u \perp v$.

Theorem. Let $(V, V^+, |\cdot|, \|\cdot\|)$ be a norm complete absolute order smooth ∞-normed space. If U is a total ∞-orthonormal set in V^+, then V is isometrically order isomorphic to $c_0(I)$ where I is the cardinality of U.

Geometric orthogonality in ordered vector spaces

Theorem. Let (V, e) be an order unit space. The following two sets of conditions are equivalent:

1. V is an absolutely ordered vector space in which $\perp = \perp_\infty^a$ on V^+.
2. V satisfies the following conditions:
   1. For each $u \in V$, the exists a unique pair $u^+, u^- \in V^+$ with $u^+ \perp_\infty^a u^-$ such that $u = u^+ - u^-$;
      Set $|u| := u^+ + u^-$.
   2. If $u, v, w \in V^+$ with $u \perp_\infty^a v$ and $u \perp_\infty^a w$, then we have $u \perp_\infty^a |v \pm w|$.

In this case, $(V, |\cdot|, e)$ is called an absolute order unit space.

A counter example

A counter example

Theorem. Let V be a real normed linear space. Consider $V^{(\cdot)} := V \times \mathbb{R}$ and put $V^{(\cdot)+} := \{(v, \alpha) : \|v\| \leq \alpha\}$. Then $(V^{(\cdot)}, V^{(\cdot)+})$ becomes a real ordered space. For $(v, \alpha) \in V^{(\cdot)}$, we define

$$|(v, \alpha)| = \begin{cases} (v, \alpha), & \text{if } (v, \alpha) \in V^{(\cdot)+} \\ -(v, \alpha), & \text{if } (v, \alpha) \in -V^{(\cdot)+} \\ \left(\frac{\alpha}{\|v\|} v, \|v\| \right), & \text{if } (v, \alpha) \notin V^{(\cdot)+} \bigcup -V^{(\cdot)+}. \end{cases}$$

Then $(V^{(\cdot)}, V^{(\cdot)+}, |\cdot|)$ is an absolutely ordered space if and only if V is strictly convex.

1. $|u| = u$ if and only if $u \in V^+$.
2. $|u| \pm u \in V^+$.
3. $|ku| = |k||u|$ for all $k \in \mathbb{R}$.
4. If $|u - v| = u + v$ and $|u - w| = u + w$, then $|u - |v \pm w|| = u + |v \pm w|$.

1. $|u| = u$ if and only if $u \in V^+$.
2. $|u| \pm u \in V^+$.
3. $|ku| = |k||u|$ for all $k \in \mathbb{R}$.
4. If $|u - v| = u + v$ and $|u - w| = u + w$, then $|u - |v \pm w|| = u + |v \pm w|$.
5. If $|u - v| = u + v$ and if $0 \leq w \leq v$, then $|u - w| = u + w$.

A counter example

Let V be a strictly convex normed linear space and consider the corresponding absolutely ordered space $(V^{(\cdot)}, V^{(\cdot)+}, |\cdot|)$ is obtained by adjoining an order unit to V. We denote it by $V^{(\infty)}$.

For $(u,\alpha),(v,\beta) \in V^{(\infty)}$, we define

$$(u,\alpha)\circ(v,\beta) := \left(\alpha v + \beta u, \alpha\beta + \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)\right).$$

Theorem. The binary operation $\circ$ is bilinear in $V^{(\infty)}$ if and only if V is a Hilbert space. In this case, $V^{(\infty)}$ is unitally Jordan isomorphic to a unital JC-algebra.

A counter example

Let V be a strictly convex normed linear space and consider the corresponding absolutely ordered space $(V^{(\cdot)}, V^{(\cdot)+}, |\cdot|)$ is obtained by adjoining an order unit to V. We denote it by $V^{(\infty)}$.

For $(u, \alpha), (v, \beta) \in V^{(\infty)}$, we define

$$(u, \alpha) \circ (v, \beta) := \left(\alpha v + \beta u, \alpha\beta + \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)\right).$$

Theorem. The binary operation $\circ$ is bilinear in $V^{(\infty)}$ if and only if V is a Hilbert space. In this case, $V^{(\infty)}$ is unitally Jordan isomorphic to a unital JC-algebra.

What is in the future?

Thank you very much for your attention!

References

(RK51) R. V. Kadison, *Order Properties of Bounded Self-Adjoint Operators*, Proc. A. M. S., **2(3)** (1951), 505–510.

(AK14) A. K. Karn, *Orthogonality in ℓ_p-spaces and its bearing on ordered normed spaces*, Positivity **18** (2014), 223–234.

(AK16) A. K. Karn, *Orthogonality in C^*-algebras.* Positivity **20** (2016), 607–620.

(AK18) A. K. Karn, *Algebraic orthogonality and commuting projections in operator algebras.* Acta. Sci. Math. (Szeged), **84** (2018), 323–353.

(AK21) A. K. Karn, *A generalization of spin factors*, Acta. Sci. Math. (Szeged), **87** (2021) 551–569.

(SS51) S. Sherman, *Order in operator algebras*, Amer. J. Math., **73(1)** (1951), 227-232.