

STEIN MANIFOLDS AND HOLOMORPHIC  
MAPPINGS 2018

17-21.9.2018



# Complete complex hypersurfaces in the ball

Antonio, Alarcón

Universidad de Granada

It was recently proved by Globevnik that, for any integer  $n \geq 2$ , the open unit ball  $\mathbb{B}_n$  of  $\mathbb{C}^n$  admits smooth complete closed complex submanifolds of arbitrary positive codimension, thereby settling in an optimal way the question posed by Yang in 1977 whether there are complete bounded complex submanifolds in a complex Euclidean space. Globevnik's hypersurfaces are given implicitly; in fact, he constructed a holomorphic function  $f: \mathbb{B}_n \rightarrow \mathbb{C}$  all of whose level sets are complete, so proving that  $\mathbb{B}_n$  carries a holomorphic foliation by complete closed complex hypersurfaces: the one formed by the family of level sets of the function  $f$ .

On the other hand, Serre proved in 1953 that every closed complex hypersurface  $V$  in a Stein manifold  $X$  with  $H^2(X; \mathbb{Z}) = 0$  is a level set of a holomorphic function on  $X$ . Much more recently, Forstnerič proved that if the hypersurface  $V$  is smooth, then it admits a holomorphic defining function on  $X$  with no critical points, and hence its level sets form a nonsingular holomorphic foliation of  $X$  by smooth closed complex hypersurfaces.

Motivated by the mentioned results, we proved that

*if a closed complex hypersurface  $V$  in  $\mathbb{B}_n$  ( $n \geq 2$ ) is complete, then it is a level set of a holomorphic function  $f$  on  $\mathbb{B}_n$  all of whose level sets are complete; if  $V$  is smooth, then  $f$  can be chosen with no critical points.*

This shows that every complete closed complex hypersurface in  $\mathbb{B}_n$  is a level set of a holomorphic function on  $\mathbb{B}_n$  as those constructed by Globevnik, thereby establishing a converse to his aforementioned existence result. In this lecture we shall discuss a general version of the above result dealing with smooth closed complex submanifolds of  $\mathbb{B}_n$  of arbitrary pure codimension. In particular, we shall show that for any pair of integers  $n$  and  $q$  with  $1 \leq q < n$  there is a nonsingular holomorphic foliation of  $\mathbb{B}_n$  by smooth complete closed complex submanifolds of pure codimension  $q$  and, what is the main point, that, under necessary assumptions, every submanifold with these properties can be embedded into such a foliation.

The main tool in the proof is an Oka-Weil-Cartan type theorem for holomorphic submersions from Stein manifolds to complex Euclidean spaces which was obtained by Forstnerič in 2003.

---

## Wild and tame sets

Rafael B., Andrist

American University of Beirut

Abstract: The complex-affine space  $\mathbb{C}^n$  has a very rich group of holomorphic automorphisms in dimension  $n \geq 2$ , in particular one can find holomorphic automorphisms with prescribed values in finitely many points. However, for infinite discrete sets, the situation is different, as observed by Rosay and Rudin in 1988: On so-called tame sets, it is possible to prescribe (tame) values, but not on so-called wild sets. While the original definition depends on coordinates, we give a coordinate-free description that extends to arbitrary complex manifolds, and investigate the interesting properties of such manifolds that contain both wild and tame sets. Joint work with Riccardo Ugolini.

---

## Superforms, Supercurrents, Riemannian Geometry and Minimal manifolds

Bo, Berndtsson

Chalmers University of Technology

Lelong's theory of positive closed forms and currents is an important tool in complex analysis and complex geometry. We will discuss a variant of this theory that applies to real submanifolds of Euclidean space. The basic idea is to associate to any such submanifold a 'supercurrent'. This is an elaboration of work of Lagerberg who studied supercurrents associated to tropical varieties. The supercurrent associated to a submanifold of Euclidean space reflects the induced Riemannian geometry and basic geometric objects like the Riemann curvature tensor are naturally interpreted as superforms. We will apply the formalism in particular to minimal manifolds and show how Lelong's technique to estimate the volume of analytic varieties can be generalized to minimal manifolds, yielding among other things a generalization of a recent theorem of Brendle and Hung.

---

# Is it converging non-tangentially?

Filippo, Bracci

Universit di Roma "Tor Vergata"

Let  $D$  be a simply connected domain in the complex plane,  $R$  a continuous curve in  $D$  landing at the boundary of  $D$  and  $f$  a Riemann map from the unit disc to  $D$ . It is well known that  $f^{-1}(R)$  lands to a point  $x$  on the boundary of the unit disc. The question this talks concerns on is: does one can find (useful) geometric conditions on  $D$  that guarantees that  $f^{-1}(R)$  converges non-tangentially (or tangentially) to  $x$ ? I will provide an answer to this question based on recent works with H. Guassier, M. Contreras, S. Diaz-Madrigal and A. Zimmer. Such a question is particularly interesting in studying dynamics of orbits of continuous semigroups of holomorphic self-maps of the unit disc (or more generally iterates of holomorphic self-maps of the unit disc). Indeed, in such case, if there is no fixed points in the unit disc, the orbits are all landing at a point of the boundary (the Denjoy-Wolff point) and the question is about the type of convergence. Every continuous semigroup of holomorphic self-maps of the unit disc admits an essential unique holomorphic model, that it, it is conjugated via a Riemann map to the dynamical system  $z \rightarrow z + it$ ,  $t \geq 0$ , where  $z$  lives in a simply connected domain  $D$  starlike at infinity (image of the intertwining Riemann map). I will explain that the orbit converge non-tangentially to the Denjoy-Wolff point if and only if the  $D$  is quasi-symmetric with respect to oneand hence anyvertical axis. This is also equivalent to the condition that oneand hence everyorbit of the semigroup is a quasi-geodesic in the sense of Gromov. A similar result holds for tangential convergence. The proof is based on new interplays between the Euclidean geometry of the domain  $D$  and the hyperbolic geometry, via the Gromov hyperbolicity theory and new results of localization of the hyperbolic distance. This also brings interesting simple consequences: for instance, if  $D$  contains a vertical angle, then the convergence is non-tangential, and this allows to construct pathological examples of semigroups with orbits that converges non non-tangentially but not tangentially, or non-tangentially but oscillating to the Denjoy-Wolff point.

---

## Foliations, Levi-flat hypersurfaces and their normal bundle

Judith, Brinkschulte

Universitt Leipzig

Levi-flat real hypersurfaces are foliated by complex hypersurfaces. They play an important role in classical complex analysis related to the Levi problem, foliations and complex dynamics: Classical nontrivial examples of Levi-flat hypersurfaces were described by Grauert as tubular neighborhoods of the zero section of a generically chosen line bundle over a non-rational Riemann surface. In these examples, the Levi-flat hypersurfaces arise as the boundary of a pseudoconvex domain admitting only constant holomorphic functions. On the other hand, there are also examples of complex surfaces that can be cut into two Stein (or even hyperconvex) domains

along smooth Levi-flat real hypersurface. These examples above show that Levi-flat hypersurfaces can be of quite different nature and therefore explain a certain interest in the classification of compact Levi-flat real hypersurfaces.

On the other hand, the study of Levi-flat real hypersurfaces is related to basic questions in dynamical systems and foliation theory: Levi-flats arise as stable sets of holomorphic foliations, and a real-analytic Levi-flat real hypersurface extends to a holomorphic foliation leaving the real hypersurface invariant.

Within the program of classifying compact Levi-flat real hypersurfaces, we will discuss curvature properties of the normal bundle to the Levi foliation, thereby presenting new non-existence results for Levi-flat CR submanifolds.

---

## Groups associated with holomorphic mappings

John P., D'Angelo

University of Illinois at Urbana-Champaign

Given a holomorphic map  $f : \Omega_1 \rightarrow \Omega_2$  between domains in complex spaces (of typically different dimensions), one can associate five groups to this map. These groups were introduced by Ming Xiao and myself. After discussing this situation in general and giving some examples, I make the assumptions that the domains are balls and that the map is proper. I show how these groups are related to early work of Forstneric. I then sketch proofs of several results. Let  $\Gamma_f$  denote the source group. We prove for a rational proper map that  $\Gamma_f$  is non-compact if and only if the map is a linear fractional transformation. Given a finite subgroup  $G$  of the source automorphism group, we show that there is a rational proper map  $f$  between balls for which  $\Gamma_f = G$ . We characterize monomial proper maps as those whose source group contains a conjugate of the torus, and orthogonal sums of tensor products as those whose source group contains a conjugate of the unitary group. If time permits, I will discuss a regularity result; if the Lie algebra of the source group of a not necessarily rational map has a certain property, then the map is actually a polynomial.

---

## On proper holomorphic discs

Barbara, Drinovec Drnovšek

University of Ljubljana

The image of a proper holomorphic map from the unit disc in  $\mathbb{C}$  is an analytic subvariety of the ambient space which is called a proper holomorphic disc. We will give an overview of the existence results of proper holomorphic discs with various additional properties. We will present the Riemann-Hilbert technique, which is frequently used in their constructions and then explain some of its recent applications. Furthermore, we will show that for certain weakly pseudoconvex domains  $D \subset \mathbb{C}^n$  there is a proper holomorphic disc in  $\mathbb{C}^n$  which intersects  $\bar{D}$  exactly at a given point of finite 1-type. This is joint work with Marko Slapar.

---

## Lagrangian skeleta of Stein manifolds

Yakov, Eliashberg  
Stanford University

I will discuss in the talk the Stein version of a joint work with D. Alvarez-Gavela, D. Nadler and L. Starkston. The main result asserts that any Stein manifold  $X$  admits a stratified subset  $C$  with standard so-called arboreal singularities and a non-negative exhausting plurisubharmonic function  $\phi$  such that  $C = \{\phi = 0\}$  and  $\phi$  is strictly plurisubharmonic and has no critical points outside of  $C$ .

---

## Entire transcendental functions in one complex dimension

John Erik, Fornæss  
NTNU

I will discuss some recent results on entire functions in one complex dimension.

---

## Finding domains of holomorphy by $C^0$ -isotopy

Robert E., Gompf  
University of Texas at Austin

We will discuss a general technique for finding domains of holomorphy in Stein surfaces, or more generally, finding Stein open subsets of complex surfaces. This begins with Eliashberg's 1990 paper constructing Stein manifolds via their plurisubharmonic Morse functions. Implicit in that paper, and proved almost explicitly by Forstnerič within a stronger theorem, is an ambient version: An open subset  $U$  of a complex manifold of (complex) dimension  $n > 2$  is smoothly isotopic (homotopic through embeddings) to a Stein manifold if and only if it admits an exhausting Morse function whose critical points have index at most  $n$ . An analogous theorem holds when  $n = 2$ :  $U$  is isotopic to a Stein surface if and only if it is diffeomorphic to a Stein surface whose complex structure is homotopic (through almost-complex structures) to the original structure on  $U$ . This allows us to find Stein open subsets with great flexibility. For example, there are many pseudoconvex 3-manifolds in  $\mathbb{C}^2$  that cut out domains of holomorphy that are contractible or homotopy equivalent to  $S^2$  (the latter answering a question raised by Forstnerič).

The difficulty with the above theorem is that it requires us to determine whether  $U$  is diffeomorphic to a Stein surface, but Eliashberg's characterization of Stein manifolds can be difficult to apply when  $n = 2$ . We can circumvent this problem at the expense of allowing infinite topology. We allow the isotopy to be *topological*, i.e., continuous but not necessarily differentiable. We obtain a closer analog of the high-dimensional theorem:  $U$  is topologically isotopic to a Stein surface if and only if  $U$  is *homeomorphic* to a manifold with a Morse function as above. The main additional ingredient here is Freedman's technology for studying topological 4-manifolds, which can be made compatible with Eliashberg's construction. Replacing

2-handles by *Casson handles* preserves the homeomorphism type of  $U$ , but can change the diffeomorphism type to be more amenable to Stein theory. For example,  $S^2 \times \mathbb{R}^2$  admits no Stein structure until we replace its 2-handle by a suitable Casson handle. Then it embeds in  $\mathbb{C}^2$  as a domain of holomorphy, and any 2-knot can be realized in this manner.

A deeper dive into Freedman theory reveals Stein neighborhoods with richer structure. As a simple example, inside a complex line bundle  $U$  over a surface (of real dimension 2), let  $U_t$  be the open disk bundle of radius  $t$ . While these may not admit Stein structures, every topological embedding of  $U$  in a complex surface is topologically isotopic to one such that for each  $t > 0$  in the standard Cantor set,  $U_t$  maps onto a Stein open subset. These subsets form an uncountable neighborhood system of the image of the 0-section, which is a topologically embedded surface that is smooth and totally real except at one nonsmooth point. The embedding can be chosen so that these Stein neighborhoods represent infinitely many diffeomorphism types, uncountably many if  $c_1(U) \leq 0$ , even though they are topologically isotopic to each other and homeomorphic to  $U$ . Any tamely, topologically embedded 2-complex in a complex surface admits a neighborhood with similar structure after a topological isotopy. It follows that every domain of holomorphy in  $\mathbb{C}^2$  is topologically isotopic to others representing uncountably many diffeomorphism types. Thus, we obtain uncountably many exotic  $\mathbb{R}^4$  domains of holomorphy in  $\mathbb{C}^2$ .

Although each Stein neighborhood  $U_t$  has infinite smooth topology, it is bounded by a compact 3-manifold that is tamely topologically embedded. Such an embedded 3-manifold is pseudoconvex in a topological sense that may deserve further study. These 3-manifolds share some of the characteristics of Stein fillable contact 3-manifolds such as a canonical orientation, a homotopy class of plane fields, and a propensity to cut out Stein surfaces, yet they are more plentiful. Every closed 3-manifold admits such an embedding into a compact complex surface, with flexibility in the choice of homotopy class of plane fields. In fact, for homology 3-spheres, all such classes can be realized. Looking at the outside of such embedded 3-manifolds leads to the notion of a topologically pseudoconvex complex structure. Uncountably many exotic smoothings of  $\mathbb{R}^4$  admit such structures.

## Elliptic characterization and localization of Oka manifolds

Yuta, Kusakabe

University of Osaka

Gromov's Oka principle gives a sufficient condition, called ellipticity, for a complex manifold to be Oka. It is a well-known problem whether the Oka property can be characterized by ellipticity or its variants. There is another ellipticity condition introduced by Gromov, which is called Condition  $Ell_1$ . We prove that Condition  $Ell_1$  characterizes Oka manifolds. The proof uses the techniques developed in the proof of Forstnerič's Oka principle. This characterization gives affirmative answers to Gromov's conjectures. As another application, we establish the localization principle for Oka manifolds, which gives new examples of Oka manifolds.

## A general approach to the Oka principle

Frank, Kutzschebauch

University of Bern

The proof of an Oka principle is in all known instances very difficult and long. We present a unifying approach of proof to different Oka principles due to Grauert, Grauert-Kerner, Ramspott, Forster-Ramspott, Hayes, Gromov-Forstneric-Prezelj as well as equivariant versions due to Heinzner-Kutzschebauch, Kutzschebauch-Larusson-Schwarz as well as Oka principles for equivariant isomorphisms over the categorical quotient due to the same authors. The main result is a homotopy theorem which sort of axiomatizes the proof of Oka principles, thus clearly distinguishing between a difficult homotopy theoretic machinery and results (mainly of complex analytic nature) specific for the Oka principle in question.

---

## Convergence and divergence of formal maps between CR manifolds

Bernhard, Lamel

Universitt Wien

We discuss the convergence problem for formal holomorphic maps taking a real-analytic submanifold in some complex Euclidean space into a real-analytic set in some other complex space, i.e. the question whether every such formal map necessarily converges. Our main result shows that if there exists a divergent formal map, then the target set has to contain some complex variety. The technique used to prove this result also solves convergence problems for sources and targets foliated by complex manifolds, and can be adapted to treat other regularity questions for CR maps.

---

## Legendrian curves in projectivised cotangent bundles

Finnur, Larusson

University of Adelaide

I will report on new joint work with Franc Forstnerič about introducing ideas and methods from Oka theory into complex contact geometry. This is a new research area in which only a few papers have been published so far (by Alarcón, Forstnerič, López, and myself). More specifically, we study holomorphic Legendrian curves in the projectivised cotangent bundle  $X = \mathbb{P}T^*Z$  of a complex manifold  $Z$  of dimension at least 2. Such a manifold  $X$  carries a natural complex contact structure. We prove several approximation and general position theorems, as well as a few versions of the h-principle, for holomorphic Legendrian curves in  $X$ . I will start with a very brief introduction to complex contact geometry.

---

# New wandering domains

Han, Peters

University of Amsterdam

In recent work of Astorg et al it was shown that polynomial maps in dimensions 2 and higher can have wandering domains; a striking contrast with the one-dimensional setting. In this work an essentially unique construction of such a domain was given.

In two recent works, one with David Hahn (PhD student in Trondheim with John Erik Fornæss) and one with Luka Boc-Thaler (Ljubljana and Rome) and Matthieu Astorg (Orleans) we used quite similar techniques to construct wandering domains with different geometry and dynamical behavior. As a result of our work we also obtain a better understanding of the methods used in the first construction.

In this talk I will focus on the work with Boc-Thaler and Astorg, where we use a Lavaurs map with a Siegel disk instead of a Lavaurs map with an attracting fixed point. Since the stability of such a disk is much more subtle than that of a contracting region, our proof requires more precise estimates for the two-dimensional system, forcing changes that actually simplify parts of the proof. Our result also requires a thorough investigation of one-dimensional non-autonomous dynamical systems given by perturbations of Siegel disks.

---

## Automorphisms of $\mathbb{C}^2$ with an invariant Fatou component biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ .

Jasmin, Raissy

Universite Toulouse III

I will present the construction of a family of automorphisms of  $\mathbb{C}^2$  having an invariant, non-recurrent Fatou component biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$  and which is attracting, in the sense that all the orbits converge to a fixed point on the boundary of the component. Such component is obtained by globalizing, using a result of Forstneric, a local construction, which allows to create a global basin of attraction for an automorphism, and a Fatou coordinate on it. The Fatou coordinate turns out to be a fiber bundle map on  $\mathbb{C}$ , whose fiber is  $\mathbb{C}^*$ , forcing the global basin to be biholomorphic to  $\mathbb{C} \times \mathbb{C}^*$ . The most subtle point is to show that such a basin is indeed a Fatou component. This is done exploiting Pöschels results about existence of local Siegel discs and suitable estimates for the Kobayashi distance. Since attracting Fatou components are Runge, it turns out that this construction gives also an example of a Runge embedding of  $\mathbb{C} \times \mathbb{C}^*$  in  $\mathbb{C}^2$ . Moreover, this example shows an automorphism of  $\mathbb{C}^2$  leaving invariant two analytic discs intersecting transversally at the origin. (This is a joint work with Filippo Bracci and Berit Stensones).

## A countable characterisation of smooth algebraic plane curves

Tuyen, Truong  
University of Oslo

Given a smooth algebraic curve  $X$  in  $\mathbb{C}^3$ , I will present a way to construct a sequence of algebraic varieties (whose ideals are explicitly determined from the ideal defining  $X$ ), whose solution set is non-empty iff the curve  $X$  can be algebraically embedded into  $\mathbb{C}^2$ .

---

## A Biased Survey of $L^2$ Extension

Dror, Varolin  
Stony Brook University

The problem of extension of holomorphic functions from analytic subsets has many variants. One of the most important variants, is extension with  $L^2$  estimates, and even among those there are different versions. In this talk I will give a survey of a particular corner of the subject that I have worked in.

---

## Tame discrete subsets in Complex Algebraic Groups

Jörg Winkelmann  
Ruhr-Universität Bochum

For discrete subsets in  $\mathbb{C}^n$  the notion of being “tame” was defined by Rosay and Rudin. A discrete subset  $D \subset \mathbb{C}^n$  is called “tame” if and only if there exists an automorphism  $\phi$  of  $\mathbb{C}^n$  such that  $\phi(D) = \mathbb{N} \times \{0\}^{n-1}$ .

We are interested in similar notions for complex manifolds other than  $\mathbb{C}^n$ .

Therefore we propose a new definition, show that it is equivalent to that of Rosay and Rudin if the ambient manifold is  $\mathbb{C}^n$  and deduce some standard properties.

To obtain good results, we need some knowledge on the automorphism group of the respective complex manifold. For this reason we get our best results in the case where the manifold is biholomorphic to a complex Lie group.

**Definition:** *Let  $X$  be a complex manifold. An infinite discrete subset  $D$  is called (weakly) tame if for every exhaustion function  $\rho : X \rightarrow \mathbb{R}^+$  and every map  $\zeta : D \rightarrow \mathbb{R}^+$  there exists an automorphism  $\phi$  of  $X$  such that  $\rho(\phi(x)) \geq \zeta(x)$  for all  $x \in D$ .*

Andrist and Ugolini have proposed a different notion, namely the following:

**Definition:** *Let  $X$  be a complex manifold. An infinite discrete subset  $D$  is called (strongly) tame if for every injective map  $f : D \rightarrow D$  there exists an automorphism  $\phi$  of  $X$  such that  $\phi(x) = f(x)$  for all  $x \in D$ .*

It is easily verified that “strongly tame” implies “weakly tame”. For  $X \simeq \mathbb{C}^n$  both tameness notions coincide with each other and with the tameness notion of Rosay and Rudin.

In the sequel, unless explicitly stated otherwise, tame always means weakly tame.

For tame discrete sets in  $\mathbb{C}^n$  in the sense of Rosay and Rudin, the following facts are well-known:

1. Any two tame sets are equivalent.
2. Every discrete subgroup of  $(\mathbb{C}^n, +)$  is tame as a discrete set.
3. Every discrete subset of  $\mathbb{C}^n$  is the union of two tame ones.
4. There exist non-tame subsets in  $\mathbb{C}^n$ .
5. Every injective self-map of a tame discrete subset of  $\mathbb{C}^n$  extends to a biholomorphic self-map of  $\mathbb{C}^n$ .
6. If  $v_k$  is a sequence in  $\mathbb{C}^n$  with  $\sum_{k=1}^{\infty} \frac{1}{\|v_k\|^{2n-1}} < \infty$ , then  $\{v_k : k \in \mathbb{N}\}$  is a tame discrete subset.

We prove the following.

**Theorem:** *Let  $G$  be a complex linear algebraic group whose character group is trivial, i.e., such that there is no non-constant morphism of algebraic groups from  $G$  to the multiplicative group  $\mathbb{C}^*$ .*

*Let  $D$  be a discrete subset.*

*Then the following conditions are equivalent:*

1.  $D$  is (weakly) tame.
2.  $D$  is strongly tame.
3. There exists a biholomorphic self-map  $\phi$  of  $G$  and a unipotent subgroup  $U \subsetneq G$  such that  $\phi(D) \subset U$ .

*Furthermore:*

1. Any two tame discrete sets are equivalent.
2. Every discrete set may be realized as the union of two tame discrete sets.
3. If  $D$  is tame, then  $G \setminus D$  is an Oka manifold.
4.  $SL_n(\mathbb{Z})$  is a tame discrete subset of  $SL_n(\mathbb{C})$ .

# **An Embedding of the unit ball in $\mathbb{C}^3$ whose image is not Runge in any strictly larger domain that contains it.**

Erlend F., Wold  
University of Oslo

We will explain the construction of an embedding as described in the title, and explain how it settles a central problem in geometric function theory in higher dimensions (in dimension three and higher).

---

## **Chern forms of metrics with analytic singularities**

Elizabeth, Wulcan  
Chalmers University of Technology

In a recent paper Lärkäng, Raufi, Ruppenthal, and Sera constructed Chern forms, or rather currents,  $c_k(E, h)$  associated with a Griffiths positive singular metric  $h$  on a holomorphic vector bundle  $E$ , that is non-degenerate outside a variety of codimension at least  $k$ .

I will discuss a joint work with Lärkäng, Raufi, and Sera, where we define Chern forms for any  $k$  in the case when  $h$  has analytic singularities. Our construction uses a generalized Monge-Ampère operator for plurisubharmonic functions with analytic singularities, recently introduced by Andersson and me. Moreover our Chern forms coincide with the Lärkäng-Raufi-Ruppenthal-Sera  $c_k(E, h)$  when these are defined. uestions for CR maps.

---