Skeletal Polyhedra, Polygonal Complexes, and Nets

Egon Schulte
Northeastern University

Rogla 2014
**Polyhedra**  With the passage of time, many changes in point of view about polyhedral structures and their symmetry.

Platonic (solids, convexity), Kepler-Poinsot (star polygons),

*Icosahedron* $\{3,5\}$
Petrie-Coxeter polyhedra (convex faces, infinite), .....
Skeletal approach to polyhedra and symmetry!

- Branko Grünbaum (1970’s) — geometrically and combinatorially.

- Allow skew faces! Restore the symmetry in the definition of “polyhedron”! Graph-theoretical approach!

- What are the regular polyhedra in ordinary space? Answer: Grünbaum-Dress Polyhedra.

- The group theory forces skew faces and vertex-figures! General reflection groups.
Polyhedron

A polyhedron $P$ in $\mathbb{E}^3$ is a (finite or infinite) family of simple polygons, called faces, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit,
- $P$ is connected,
- each compact set meets only finitely many faces (discreteness).

All traditional polyhedra are polyhedra in this generalized sense.
Highly symmetric polyhedra $P$

- Conditions on the geometric symmetry group $G(P)$ of $P$.

- $P$ is **regular** if $G(P)$ is transitive on the flags.
  
  *Flag*: incident triple of a vertex, an edge, and a face.

- $P$ is **chiral** if $G(P)$ has two orbits on the flags such that adjacent flags are in distinct orbits.

- Other interesting classes: Archimedean (regular faces, vertex-transitive); vertex-transitive (isogonal); face-transitive (isohedral); edge-transitive (isotoxal); .......
The helix-faced regular polyhedron \( \{\infty, 3\}^{(b)} \), with symmetry group requiring the single extra relation \((R_0 R_1)^4 (R_0 R_1 R_2)^3 = (R_0 R_1 R_2)^3 (R_0 R_1)^4\).
Helix-faced polyhedron $\{\infty, 3\}^{(b)}$
Regular Polyhedra in \( \mathbb{E}^3 \)

Grünbaum (70's), Dress (1981); McMullen & S. (1997)

18 finite polyhedra: 5 Platonic, 4 Kepler-Poinsot, 9 Petrials.
(2 full tetrahedral symmetry, 4 full octahedral, 12 full icosahedral)

30 apeirohedra (infinite polyhedra). Crystallographic groups!

6 planar (3 regular tessellations and their Petrials)
12 reducible apeirohedra. Blends of a planar polyhedron and a linear polygon (segment or tessellation).

Square tessellation blended with the line segment. Symbol $\{4,4\}\#\{\}$

The square tessellation blended with a line tessellation. Each vertical column over a square is occupied by exactly one helical facet spiraling around the column. Symbol $\{4,4\}\#\{\infty\}$
12 irreducible apeirohedra.

\[
\begin{align*}
\{\infty, 4\}_{6,4} & \xrightarrow{\pi} \{6, 4|4\} & \xrightarrow{\delta} \{4, 6|4\} & \xrightarrow{\pi} \{\infty, 6\}_{4,4} \\
\sigma \downarrow & & & \downarrow \eta \\
\{\infty, 4\}_{.,*3} & \xrightarrow{\varphi_2} \{6, 6\}_{4} & \xrightarrow{\varphi_2} \{\infty, 3\}_{(a)} \\
\pi \uparrow & & & \uparrow \pi \\
\{6, 4\}_{6} & \xrightarrow{\delta} \{4, 6\}_{6} & \xrightarrow{\varphi_2} \{\infty, 3\}_{(b)} \\
\sigma \delta \downarrow & & & \downarrow \eta \\
\{\infty, 6\}_{6,3} & \xrightarrow{\pi} \{6, 6|3\}
\end{align*}
\]

\[\eta: R_0 R_1 R_0, R_2, R_1; \quad \sigma = \pi \delta \eta \pi \delta: R_1, R_0 R_2, (R_1 R_2)^2; \quad \varphi_2: R_0, R_1 R_2 R_1, R_2\]
**Breakdown by mirror vector** (for $R_0, R_1, R_2$)

<table>
<thead>
<tr>
<th>mirror vector</th>
<th>{3, 3}</th>
<th>{3, 4}</th>
<th>{4, 3}</th>
<th>faces</th>
<th>vertex-figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,1,2)</td>
<td>{6, 6</td>
<td>3}</td>
<td>{6, 4</td>
<td>4}</td>
<td>{4, 6</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>$\infty, 6}_{4,4}$</td>
<td>$\infty, 4}_{6,4}$</td>
<td>$\infty, 6}_{6,3}$</td>
<td>helical</td>
<td>skew</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>{6, 6}_4</td>
<td>{6, 4}_6</td>
<td>{4, 6}_6</td>
<td>skew</td>
<td>planar</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>$\infty, 3}_{(a)}$</td>
<td>$\infty, 4}_{,*,3}$</td>
<td>$\infty, 3}_{(b)}$</td>
<td>helical</td>
<td>planar</td>
</tr>
</tbody>
</table>

Polyhedra in the last row occur in two enantiomorphic forms. Still, geometrically regular!

**Presentations for the symmetry group are known.** Fine Schlafli symbol signifies defining relations. Extra relations specify order of $R_0 R_1 R_2$, $R_0 R_1 R_2 R_1$, or $R_0(R_1 R_2)^2$. 
How about regular polyhedra in higher dimensions?

Coxeter’s regular skew polyhedra in $\mathbb{E}^4$ (1930’s) — convex faces and skew vertex-figures:

$\{4, 4|r\} (r \geq 3), \{4, 6|3\}, \{6, 4|3\}, \{4, 8|3\}, \{8, 4|3\}$

Arocha, Bracho & Montejano (2000), Bracho (2000): regular polyhedra in $\mathbb{E}^4$ with planar faces and skew vertex-figures

McMullen (2007):
all regular polyhedra in $\mathbb{E}^4$
Chiral Polyhedra in $\mathbb{E}^3$
S., 2004, 2005

- Two flag orbits, with adjacent flags in different orbits.
- Local picture

![Local picture of chiral polyhedra]

- Maximal “rotational” symmetry but no “reflexive” symmetry! Irreflexible!
(Regularity: maximal “reflexive” symmetry.)

- No classical examples! No finite chiral polyhedra in $\mathbb{E}^3$. 
### Three Classes of Finite-Faced Chiral Polyhedra

(S₁, S₂ rotatory reflections, hence skew faces and skew vertex-figures.)

<table>
<thead>
<tr>
<th>Schlāfli</th>
<th>{6, 6}</th>
<th>{4, 6}</th>
<th>{6, 4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation</td>
<td>P(a, b)</td>
<td>Q(c, d)</td>
<td>Q(c, d)*</td>
</tr>
<tr>
<td>Param.</td>
<td>a, b ∈ ℤ, (a, b) = 1</td>
<td>c, d ∈ ℤ, (c, d) = 1</td>
<td>c, d ∈ ℤ, (c, d) = 1</td>
</tr>
<tr>
<td></td>
<td>geom. self-dual</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P(a, b)* ≅ P(a, b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special gr</td>
<td>[3, 3]⁺ × ⟨−I⟩</td>
<td>[3, 4]</td>
<td>[3, 4]</td>
</tr>
<tr>
<td>Regular cases</td>
<td>P(a, −a) = {6,6}₄</td>
<td>Q(a, 0) = {4,6}₆</td>
<td>Q(a, 0)* = {6,4}₆</td>
</tr>
<tr>
<td></td>
<td>P(a, a) = {6,6</td>
<td>3}</td>
<td>Q(0, a) = {4,6</td>
</tr>
</tbody>
</table>

Vertex-sets and translation groups are known!
$P(1, 0)$, of type $\{6, 6\}$

Neighborhood of a single vertex.
$Q(1,1)$, of type $\{4,6\}$

Neighborhood of a single vertex.
Three Classes of Helix-Faced Chiral Polyhedra
($S_1$ screw motion, $S_2$ rotation; helical faces and planar vertex-figures.)

<table>
<thead>
<tr>
<th>Schlӓfli symbol</th>
<th>${\infty, 3}$</th>
<th>${\infty, 3}$</th>
<th>${\infty, 4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Helices over</td>
<td>triangles</td>
<td>squares</td>
<td>triangles</td>
</tr>
<tr>
<td>Special group</td>
<td>$[3, 3]^+$</td>
<td>$[3, 4]^+$</td>
<td>$[3, 4]^+$</td>
</tr>
<tr>
<td>Relationships</td>
<td>$P(a, b)\varphi^2$</td>
<td>$Q(c, d)\varphi^2$</td>
<td>$Q^*(c, d)^\kappa$</td>
</tr>
<tr>
<td></td>
<td>$a \neq b$ (reals)</td>
<td>$c \neq 0$ (reals)</td>
<td>$c, d$ reals</td>
</tr>
<tr>
<td>Regular cases</td>
<td>${\infty, 3}^{(a)}$</td>
<td>${\infty, 3}^{(b)}$</td>
<td>${\infty, 4}$,*3</td>
</tr>
<tr>
<td></td>
<td>$= P(1, -1)\varphi^2$</td>
<td>$= Q(1, 0)\varphi^2$</td>
<td>self-Petrie</td>
</tr>
<tr>
<td></td>
<td>$= {6, 6}^{\varphi^2}_4$</td>
<td>$= {4, 6}^{\varphi^2}_6$</td>
<td></td>
</tr>
</tbody>
</table>

Vertex-sets and translation groups are known!
Remarkable facts

- Essentially: any two finite-faced polyhedra of the same type are non-isomorphic.
  \[ P(a, b) \cong P(a', b') \iff (a', b') = \pm(a, b), \pm(b, a). \]
  \[ Q(c, d) \cong Q(c', d') \iff (c', d') = \pm(c, d), \pm(-c, d). \]

- Finite-faced polyhedra are intrinsically (combinatorially) chiral! [Pellicer & Weiss 2009]

- Helix-faced polyhedra combinatorially regular! Combin. only 3 polyhedra! Chiral helix-faced polyhedra are chiral deformations of regular helix-faced polyhedra! [P&W 2009]

- Chiral helix-faced polyhedra unravel Platonic solids! Coverings
  \[ \{\infty, 3\} \mapsto \{3, 3\}, \quad \{\infty, 3\} \mapsto \{4, 3\}, \quad \{\infty, 4\} \mapsto \{3, 4\}. \]
Polytopes of Higher Ranks

*Regular:* McMullen 2000’s

In $\mathbb{R}^4$: 34 finite of rank 4; 14 infinite of rank 5.

*Chiral:* Bracho, Hubard & Pellicer (2014)

Examples of chiral polytopes of rank 4 in $\mathbb{E}^4$. 
Regular Polygonal Complexes in $\mathbb{E}^3$ (with D. Pellicer)

(Hybrids of polyhedra and incidence geometries. Polyhedral geometries.)

A polygonal complex $K$ in $\mathbb{E}^3$ is a family of simple polygons, called faces, such that

- each edge of a face is an edge of exactly $r$ faces ($r \geq 2$);
- the vertex-figure at each vertex is a connected graph, possibly with multiple edges;
- the edge graph of $K$ is connected;
- each compact set meets only finitely many faces (discreteness).

$K$ is regular if its geometric symmetry group $G(K)$ is transitive on the flags of $K$. 
..... The End ..... 

Thank you
Abstract

Skeletal polyhedra and polygonal complexes in 3-space are finite, or infinite periodic, geometric edge graphs equipped with additional polyhedra-like structure determined by faces (simply closed planar or skew polygons, zig-zag polygons, or helical polygons). The edge graphs of the infinite polyhedra and complexes are periodic nets. We discuss classification results for skeletal polyhedra and polygonal complexes in 3-space by distinguished transitivity properties of the symmetry group, as well as the relevance of these structures for the classification of crystal nets.