Irreducibility of configurations

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Incidence geometry

An **incidence geometry** is a triple (P, L, I) where

- P is a set of 'points',
- L is a set of 'blocks',
- I is an incidence relation between the elements in P and L.

When there are at most one block containing p_i and p_j for all pairs of points, then we call the blocks **lines**.

Incidence graph of an incidence geometry

We can use a graph to represent the incidences of points and blocks.

The **incidence graph** of the incidence structure (P, L, I) is the bipartite graph with vertex set $P \cup L$ and an edge between the vertices p and b if p is a point on b.

Combinatorial configurations

A **combinatorial** (v, b, r, k)-**configuration** is an incidence structure with v points and b lines/blocks such that

- every point appears on r lines,
- every line has k points,
- every pair of points is in at most one line, or equivalently,
- every pair of lines intersect in at most one point.

The four parameters (v, b, r, k) are redundant.
We only need the three parameters (d, r, k) with

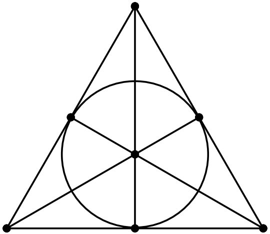
$$d := \frac{v \gcd(r, k)}{k} = \frac{b \gcd(r, k)}{r} = \frac{vr}{\mathrm{lcm}(r, k)} = \frac{bk}{\mathrm{lcm}(r, k)}.$$

Reduced parameters: (d, r, k)-configuration.

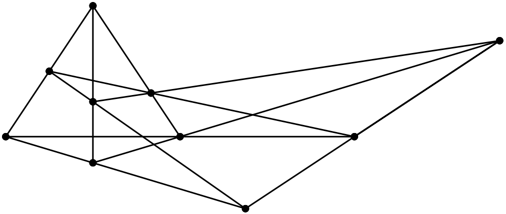
A combinatorial (v, b, r, k) configuration is also called an r-regular and k-uniform **partial linear space**.

Balanced configurations

We say that a combinatorial configuration is **balanced** if $r = k$. This implies that the number of points equals the number of lines and also, the associated integer, so $d = v = b$.



The Fano plane,
$(v, b, r, k) = (7, 7, 3, 3)$
$(d, r, k) = (7, 3, 3)$

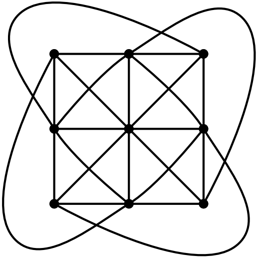


The Desargues' configuration
$(v, b, r, k) = (10, 10, 3, 3)$
$(d, r, k) = (10, 3, 3)$

Non-balanced configurations

When $r \neq k$, then $v \neq b$ and $d = \frac{v \gcd(r,k)}{k}$.

- The affine plane $AG(2, q)$ over the finite field $\mathbb{F}_q$ has parameters $(q^2, q^2 + q, q + 1, q)$ so $d = q$.
  Reduced parameters: $(q, q + 1, q)$.
- A Steiner triple systems of order v ($STS(v)$) has parameters $(v, v(v - 1)/6, (v - 1)/2, 3)$.
  Reduced parameters: $(v \gcd(v - 1, 3)/3, (v - 1)/2, 3)$.



$AG(2, 3)$ / $STS(9)$

$$
\begin{aligned}
(v, b, r, k) &= (9, 12, 4, 3) \\
(d, r, k) &= (3, 4, 3)
\end{aligned}
$$

Necessary conditions for existence of configurations

The following necessary conditions for existence of configurations are well-known.

Lemma.
Suppose that there exists a (v, b, r, k)-configuration. Then

1. $v \geq r(k-1)+1$ and $b \geq k(r-1)+1$, and
2. $vr = bk$.

We say that parameters satisfying these conditions are **admissible**.

What about sufficient conditions?

Sufficent conditions

- When $r = 3$, the necessary conditions are sufficent [Gropp (1994)].
- When $r = 4$, it is conjectured that the necessary conditions are sufficient [Gropp (2001)].
- When $r = 5$, the necessary conditions are not sufficent. Sufficient conditions are not known for $k > r$.
- In general sufficient conditions are not known.

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Augmenting balanced configurations [Martinetti, 1886]

Given a $(v, v, 3, 3)$-configuration, add a point and a line to construct a $(v+1, v+1, 3, 3)$-configuration.

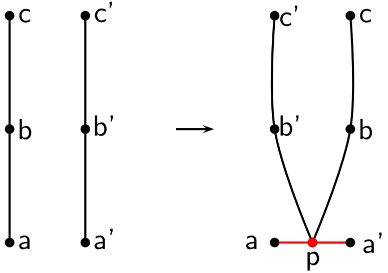
How?

Assume that there are two parallel lines $\{a, b, c\}$ and $\{a', b', c'\}$, with a and a' noncollinear.

Add a point p and replace the two parallel lines with the lines $\{p, b, c\}$, $\{p, b', c'\}$, $\{p, a, a'\}$.

The result is a $(v+1, v+1, 3, 3)$-configuration.

The Martinetti augmentation



Reduction of configurations I [Martinetti, 1886]

A configuration is called irreducible if it cannot be constructed from a smaller configuration using the augmentation construction.

Theorem. [Martinetti- Boben]
The irreducible configurations à la Martinetti are:

- Cyclic configurations with base line $\{0, 1, 3\}$ (starting with the Fano plane).
- Three infinite families $T_1(n)$, $T_2(n)$, $T_3(n)$, on $10n$ points. The smallest configuration in $T_1(n)$ is the Desargues' configuration.
- The Pappus' configuration.

Reduction of configurations II [Carstens et al., 2001]

Given a $(v, v, 3, 3)$-configuration, remove a point and a line to construct a $(v-1, v-1, 3, 3)$-configuration.

How?

A complicated family of several Martinetti-like reductions defined on the incidence graph.

Their goal was to show that the only irreducible configuration was the Fano plane.

Unfortunately, in 2005, Ravnik used a computer to show that they failed to reduce at least the Desargues' configuration.

Reduction of configurations III [Boben, 2005]

In the incidence graph of a $(v, v, 3, 3)$-configuration, remove a point-vertex p and a line-vertex ℓ and connect their neighbors so that the result is an incidence graph of a $(v-1, v-1, 3, 3)$-configuration.

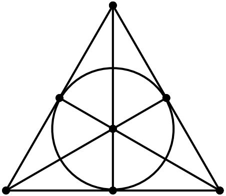
The incidence graph of a $(v, v, 3, 3)$-configuration is a bipartite cubic graph of girth at least 6. The result is a $(v-1, v-1, 3, 3)$-configuration.

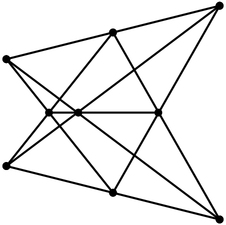
Martinetti's reduction is a special case of this reduction.

Irreducible configurations

Theorem. [Boben (2005)] Boben's irreducible configurations are:

- The Fano plane.
- The Pappus' configuration.





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Augmenting balanced configurations with $r = k \geq 3$

Theorem.

Assume there is

- a (balanced) (d, k, k)-configuration (P, L, I) with k points $Q \subseteq P$ and k lines $M \subseteq L$, and
- a bijection $f : Q \to M$ defined as follows:
  - the image of a point $q \in Q$ is a line $f(q) \in M$ through that point,
  - two points $q, q' \in Q$ can be collinear only on the line $f(q)$ or $f(q')$,
  - two lines $m, m' \in M$ can meet only in the point $f^{-1}(m)$ or $f^{-1}(m')$.

Then there is a $(d + 1, k, k)$-configuration constructed from C through an augmentation procedure.

Augmenting balanced configurations with $r = k \geq 3$

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Then there is a $(d + 1, k, k)$-configuration constructed from C through an augmentation procedure.

- Disconnect all incidences $(q, f(q)) \in I$.
- Add a new line ℓ and the incidences (q, ℓ) for all points $q \in Q$.
- Add a new point p and the incidences (p, m) for all lines $m \in M$.

The result is a configuration with parameters $(d + 1, k, k)$.

Example: Augmenting $(v, v, 3, 3,)$-configurations

Lemma. Every $(v, v, 3, 3)$-configuration admits an augmentation.

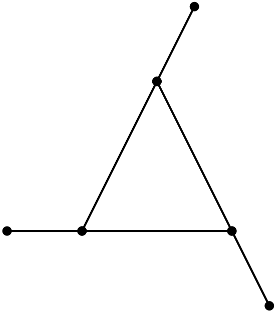
Proof.

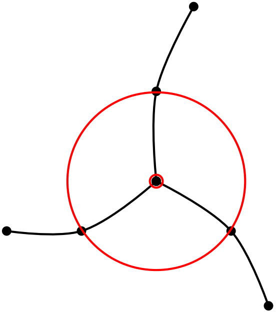
- If the configuration contains a triangle, take Q and M the three points and the three lines in the triangle.
- If the configuration contains no triangle, there are still three points a, b, c such that (a, b) and (b, c) are collinear on the two lines A, B, and a third line C through a not meeting A nor B. □

This implies the following well-known result.

Corollary. There is a $(v, v, 3, 3)$ configuration whenever the parameters are admissible.

Augmenting $(v, v, 3, 3)$-configurations





Deficiency of a configuration

The **distance** between two points is the number of lines in a “shortest path” between them.

In a (d, r, k)-configuration all points have the same number $r(k-1)$ of points at distance 1.

The **deficiency** of a configuration is the number of points at distance at least 2 from a given point.

Augmenting $(v, v, 4, 4)$-configurations

Lemma. A $(d, 4, 4)$-configuration admits an augmentation if and only if it has deficiency at least 1.

Proof.

- If deficiency is 0 then it is the finite projective plane of order 3, which is not augmentable.
- If deficiency is ≥ 1 then there are always points a, b, c, d such that the pairs (a, b), (b, c), (c, d) are collinear on the lines A, B, C, and the pairs (a, c), (b, d) are at distance at least two. The forth line D can be taken as the line through a and d if there is such a choice of points and lines. Otherwise D can be taken through d such that it does not meet A, B, C. □

There are $(v, 4, 4)$-configuration with deficiency 0 and 1, so we get the following well-known result.

Corollary. There is a $(v, v, 4, 4)$-configuration whenever the parameters are admissible.

Augmenting configurations with $r, k \geq 3$

Theorem. Let $t = rk/\gcd(r,k)$. Assume there is

- a (d,r,k)-configuration (P,L,I) with t points $Q \subseteq P$ and t lines $M \subseteq L$, and
- a bijection $f : Q \to M$ defined as follows:
  - the image of a point $q \in Q$ is a line $f(q) \in M$ through that point,
  - $Q = \bigcup_{i=1}^{r/\gcd(r,k)} Q_i$ such that $|Q_i| = k$, $Q_i \cap Q_j = \emptyset$, and two points $q_i, q_i' \in Q_i$ can be collinear only on the line $f(q_i)$ or $f(q_i')$,
  - $M = \bigcup_{i=1}^{k/\gcd(r,k)} M_i$ such that $|M_i| = r$, $M_i \cap M_j = \emptyset$, and two lines $m_i, m_i' \in M_i$ can meet only in the point $f^{-1}(m_i)$ or $f^{-1}(m_i')$.

Then there is a $(d+1,r,k)$-configuration constructed from C through an augmentation procedure.

Augmenting configurations with $r, k \geq 3$

Proof.

- Disconnect all incidences $(q, f(q)) \in I$.
- For each Q_i add a new line ℓ_i and the incidences (q_i, ℓ_i) for all points $q_i \in Q_i$.
- For each M_i add a new point p_i and the incidences (p_i, m_i) for all lines $m_i \in M_i$.

The result is a configuration with parameters $(d+1, r, k)$.

$$\begin{array}{lcl}(d, r, k) & \to & (d+1, r, k) \\ (v, b, r, k) & \to & (v + k/\gcd(r,k), b + r/\gcd(r,k), r, k)\end{array}$$

□

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Reduction of **balanced** configurations with $r = k \geq 3$

A **reduction** of a balanced configuration (P, L, I) is a triple (p, ℓ, f) where

- p is a point,
- ℓ is a line,
- f' is a bijection $f' : Q' \to M'$,
  where
  - $Q' = \{q : q \in \ell \text{ and } q \neq p\}$, and
  - $M' = \{m : p \in m \text{ and } m \neq \ell\}$,

  such that q is not collinear with $s \in f'(q)$ except possibly through ℓ or with p.

Now delete p and l (and their incidences) and add incidences $(q, f'(q))$ for $q \in Q'$.

A balanced configuration is **irreducible** if it does not admit a reduction.

Lemma. The reduction is the inverse operation of the augmentation.

Reduction of configurations with $r, k \geq 3$

A **reduction** of a configuration (P, L, I) is a triple (R, N, f') where

- R is a set of points,
- N is a set of lines,
- f' is a pairing between the elements of two multisets $f : Q' \to M'$, where
  - $Q' = \{q : q \in P$ and $\exists \ell \in N$ such that $q \in \ell$ and $q \notin R\}$,
  - $M' = \{m : m \in L$ and $\exists p \in R$ such that $p \in m$ and $m \notin N\}$,

  such that q is not collinear with $s \in f(q)$ except possibly through one of the lines in N or with one of the points in R.

Now delete R and N and their incidences and add incidences $(q, f'(q))$ for $q \in Q'$.

A configuration is **irreducible** if it does not admit a reduction.

Lemma. The reduction is the inverse operation of the augmentation.

Reduction of $(d, 3, 3)$-configurations

In the case $(d, 3, 3)$ this definition has the same implications as Boben's reduction.

Lemma There are only two irreducible $(d, 3, 3)$-configurations:

- The Fano plane,
- The Pappus' configuration.

The Pappus' configuration as a transversal design

What is the analog of the Pappus' configuration for other values of r and k?

The Pappus' configuration is a **resolvable transversal design**.

A transversal design $TD_\lambda(k, n)$ is a (k-uniform) incidence geometry on $v = kn$ points partitioned into k groups of n elements, such that

- any group and any block contain exactly one common point, and
- every pair of points from distinct groups is contained in exactly λ blocks.

A transversal design is **resolvable** if the line set can be partitioned in parallel classes and it is a (kn, n^2, n, k)-configuration if $\lambda = 1$.

Example: There is a resolvable $TD_1(k, n)$ whenever there is an affine plane of order n and $k \leq n$. Take the points on k lines in a parallel class and restrict the rest of the lines to these points.

Irreducibility of resolvable transversal designs

Lemma. A resolvable transversal design $TD_1(k, n)$ is irreducible if $k \geq (k + r)/\gcd(r, k) + 1$.

Proof.

- Let p be a point in $T = TD_1(k, n)$ and $\ell_1, \ldots, \ell_n$ the lines through p.
- Then $\ell_1, \ldots, \ell_n$ are in different parallel classes.
- Let ℓ be a line in T and q a point on ℓ.
- Then q is collinear with all points on the lines $\ell_1, \ldots, \ell_n$ except one on each line.
- At most $(r + k)/\gcd(r, k)$ of these incidences can be removed by a reduction and do (perhaps) not obstruct reduction.
- More than $(r + k)/\gcd(r, k)$ such incidences will obstruct reduction.

Large configurations are reducible

Lemma. A (v, b, r, k)-configuration is reducible if $b \geq 1 + r + r(k-1)(r-1) + r(k-1)^2(r-1)^2$.

However this bound is not sharp.

Thank you for listening!